If $A$ is a measurable subset of $\mathbb{R}^n$, two functions $f, g: A \to \mathbb{R}$ are said to be orthogonal with respect to a weight function $w: A \to \mathbb{R}$ if $\int_A fgw \, dm = 0$ where $m$ is the usual measure on $\mathbb{R}^n$. A familiar example is the theory of Fourier series which is based on the fact any two functions in the set $\{1\} \cup \{\cos nx, \sin nx\}_{n=1}^{\infty}$ are orthogonal on the interval $[-\pi, \pi]$. On a philosophical note, I will point out that if there is a Creator, that Being would seem to share my fascination with orthogonality. It is no exaggeration to say that quantum theory is, at root, a profound application of orthogonality. More generally, orthogonality arguments are ubiquitous in mathematical physics.

In my talk I will discuss a classical body of theory describing polynomials in one variable that are orthogonal on the interval $[-1, 1]$ with respect to various weight functions $w: [-1, 1] \to \mathbb{R}$ of a sufficiently simple kind. These include the Legendre polynomials (weight function $w(x) = 1$), the Chebyshev polynomials of the 1st and 2nd kinds (weight functions $w(x) = (1-x^2)^{-1/2}, (1-x^2)^{1/2}$ respectively), and the more general Gegenbauer and Jacobi polynomials. These polynomials find many applications in physics as well as pure mathematics. For example, the Legendre polynomials are needed to describe the spherical harmonic functions which arise from the classical partial differential equations of physics (potential equation, heat equation, wave equation, Schrödinger equation among others) whenever radial symmetry is present. There are also interesting orthogonal polynomials defined on infinite intervals. For example the Laguerre polynomials (interval $[0, \infty), w(x) = e^{-x}$) and the Hermite polynomials (interval $(-\infty, \infty), w(x) = e^{-x^2}$) have important applications in quantum mechanics.

After surveying this theory I discuss polynomials in two variables orthogonal on the unit disk in $\mathbb{R}^2$ with respect to a radially symmetric weight function. Although mathematically natural to think about, orthogonal polynomials in more than one variable have not been extensively studied, though people have found applications to lense optics and tomography. Due to radial symmetry of the weight function, the two dimensional rotation group $SO(2)$ acts on the vector space of orthogonal polynomials on the unit disk. Consider the vector space $V^n$ of polynomials in $x$ and $y$ of degree $n$. A very nice result due to Logan and Shepp in a 1975 paper demonstrates that an orthogonal basis of $V^n$, on the unit disk with respect to the simple weight function $w(x) = 1$, is obtained by rotating around the one variable polynomial $U^n(x)$ (the Chebyshev polynomial of 2nd kind). The original proof of Logan and Shepp, as well as subsequent somewhat simplified proofs, use sophisticated methods that involve Radon transforms on a certain Hilbert space. I will present a very short elementary proof I worked out that uses only Calculus 3 methods. I will briefly discuss a generalization of this result due to Shane Waldron.