SOLUTION PROBLEM OF THE MONTH, MARCH 2018

Congratulations to Piotr Laskawiec and Camila Larsson for submiting correct solutions to the March problem of the month!

Problem. A postal worker delivers mail to the twenty houses on a street block. The mailman notices that no two adjacent houses ever get mail on the same day but out of any three consecutive houses at least one gets some mail in any given day. How many different daily patterns of mail delivery are possible?

Solution

Identify mail delivering patterns with binary sequences of length 20 where 1's denote houses that receive mail and 0's represent houses that do not receive mail in any given day. Since no two adjacent houses ever get mail on the same day the sequence cannot have two consecutive 1's. Also, since out of any three consecutive houses at least one gets some mail in any given day, the sequence cannot have three consecutive 0's.

Let a_n be the number of binary sequences of length n that do not contain any of the substrings 11 and 000. We want to find the value of a_{20}.

By direct counting it is easy to see that $a_1 = 2$, $a_2 = 3$ and $a_3 = 4$. Let b_n be the number of binary sequences of length n that do not contain 11 or 000 and start with 0. Similarly, let c_n be the number of binary sequences of length n that do not contain 11 or 000 and start with 1. Clearly,

$$a_n = b_n + c_n$$

It is easy to see that

$$b_n = c_{n-2} + c_{n-1} \quad \text{and} \quad c_n = b_{n-1}.$$

It follows that

$$b_n = b_{n-2} + b_{n-3} \quad \text{and} \quad c_n = c_{n-2} + c_{n-3},$$

from which

$$a_n = a_{n-2} + a_{n-3}.$$

With this recurrence in hand, the values of a_n can be easily computed.

$a_4 = a_2 + a_1 = 3 + 2 = 5, a_5 = a_3 + a_2 = 4 + 3 = 7, a_6 = a_4 + a_3 = 5 + 4 = 9,$

$a_7 = a_5 + a_4 = 7 + 5 = 12, a_8 = a_6 + a_5 = 9 + 7 = 16, a_9 = a_7 + a_6 = 12 + 9 = 21,$

$a_{10} = a_8 + a_7 = 16 + 12 = 28, a_{11} = a_9 + a_8 = 21 + 16 = 37, \ldots, a_{20} = a_{18} + a_{17} = 265 + 200 = 465.$

Hence, there are 465 different mail delivery patterns for 20 houses.

As noticed by both Piotr and Camila this sequence

$$2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351, 465, \ldots$$

is a particular example of the Padovan sequence.