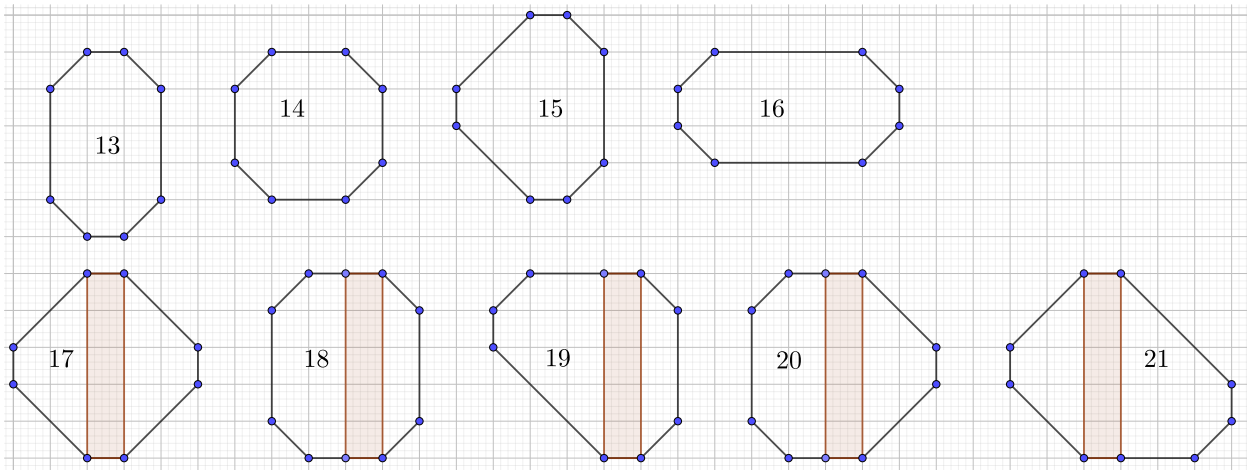
PROBLEM OF THE MONTH, OCTOBER 2019, SOLUTION

An octagon is said to be *nice* if all its interior angles are 135°, its vertices are points with integer coordinates, and its area is a positive integer. Prove that for every positive integer $A \geq 13$, there exists a nice octagon whose area is $= A$.

Solution

Here are the diagrams of nice octagons of areas 13 through 21.



Note that for the last five $(17 - 21)$ there is a shaded rectangle within each octagon exactly five units tall and one unit wide. To find a nice octagon of area 22, one would only need to widen the column in the octagon of area 17 by one unit. This would create the extra five square units. Since we have five consecutive nice octagons with this feature, it is clear that by expanding the width of these columns we can eventually achieve any area.