SOLUTION - PROBLEM OF THE MONTH, OCTOBER 2017

Congratulations to Piotr Laskawiec who found a correct solution of the October Problem!

A 4-coloring of the plane is a function $\chi : \mathbb{R}^2 \to \{\text{red, blue, green, purple}\}$, which assigns to each point in the plane exactly one of the colors red, blue, green, or purple.

Prove that for every 4-coloring of the plane, one will always have two points at distance 1 or distance $\sqrt{3}$ from each other which are colored identically.

Solution.

We suppose for the sake of contradiction that a 4-coloring of the plane exists such that no two points at distance 1 or distance $\sqrt{3}$ from each other are identically colored.

Consider the following five points:

1. $(0, 0)$, 2. $(2, 0)$, 3. $(1/2, -\sqrt{3}/2)$, 4. $(1, 0)$, 5. $(1/2, \sqrt{3}/2)$.

It is easy to see that with the exception of the pair $\{1, 2\}$, every other two points are either 1 or $\sqrt{3}$ apart. Hence, if we are to have only four colors available and we want to avoid points distance 1 or $\sqrt{3}$ apart to be colored the same, it must be that points 1 and 2 must be identically colored - see the figure below. Rotate now this point set about vertex 1 until the image of vertex 2 is one unit apart from its original position. Using the earlier reasoning for the points $\{1, 2', 3', 4', 5'\}$ it follows that points 1 and 2' must be colored the same. But then points 2 and 2' will receive the same color, which contradicts the requirement that no two points distance 1 apart can be identically colored.

One obtains a point set consisting of 9 vertices and 19 distances which are either 1 or $\sqrt{3}$, and which cannot be properly colored with only four colors.