SOLUTION - PROBLEM OF THE MONTH, OCTOBER 2017

Congratulations to *Piotr Laskawiec* who found a correct solution of the October Problem!

A 4-*coloring of the plane* is a function $\chi : \mathbb{R}^2 \to \{red, blue, green, purple\}$, which assigns to each point in the plane exactly one of the colors red, blue, green, or purple.

Prove that for every 4-coloring of the plane, one will always have two points at distance 1 or distance $\sqrt{3}$ from each other which are colored identically.

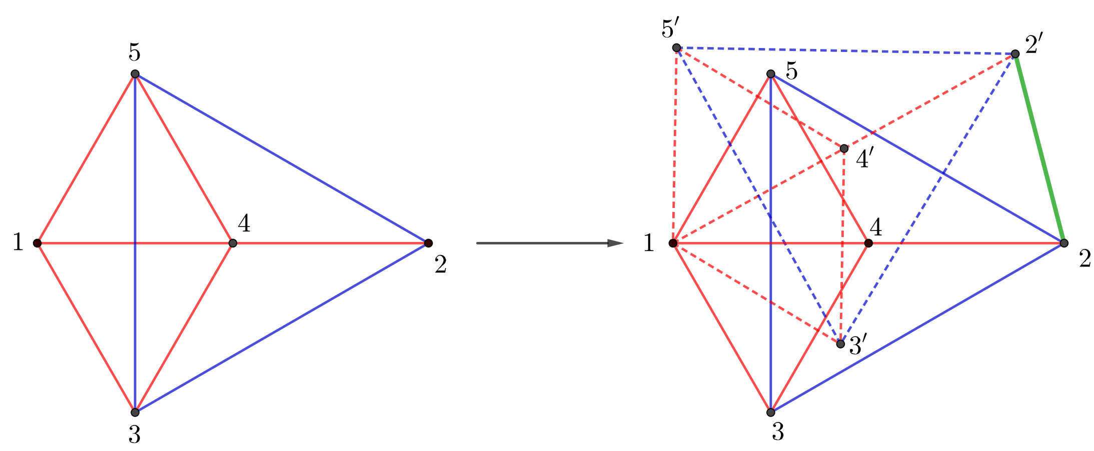
Solution.

We suppose for the sake of contradiction that a 4-coloring of the plane exists such that no two points at distance 1 or distance $\sqrt{3}$ from each other are identically colored.

Consider the following five points:

$$\mathbf{1}\,(0,0),\ \mathbf{2}\,(2,0),\ \mathbf{3}\,(1/2,-\sqrt{3}/2),\ \mathbf{4}\,(1,0),\ \mathbf{5}\,(1/2,\sqrt{3}/2).$$

It is easy to see that with the exception of the pair $\{1,2\}$, every other two points are either 1 or $\sqrt{3}$ apart. Hence, if we are to have only four colors available and we want to avoid points distance 1 or $\sqrt{3}$ apart to be colored the same, it must be that points 1 and 2 must be identically colored - see the figure below. Rotate now this point set about vertex 1 until the image of



vertex 2 is one unit apart from its original position. Using the earlier reasoning for the points $\{1,2',3',4',5'\}$ it follows that points 1 and $2'$ must be colored the same. But then points 2 and $2'$ will receive the same color, which contradicts the requirement that no two points distance 1 apart can be identically colored.

One obtains a point set consisting of 9 vertices and 19 distances which are either 1 or $\sqrt{3}$, and which cannot be properly colored with only four colors.