SOLUTION OF THE PROBLEM OF THE MONTH, SEPTEMBER 2019

Consider the integer sequence $\{x_n\}_{n\geq 0}$ given by $x_0 = 0, x_1 = 1$ and

$$x_n = 4x_{n-1} - x_{n-2}, \quad \text{for all } n \geq 2.$$

The first few terms of this sequence are

$$0, 1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316, 564719, 2107560, 7865521, 29354524, \dots$$

Find the smallest $n \geq 2$ such that x_n is a prime number, or prove that such an n does not exist.

Solution. It turns out that for every $n \geq 2$, the term x_n is composite. We will need the following statement which can be easily proved by induction.

For every $n \geq 2$ we have

$$x_{n+1}^2 - 4x_n x_{n+1} + x_n^2 = 1. \tag{1}$$

Assume that $x_{n+1} = p$, where p is a prime ≥ 3. Then, the above equality can be written as $p^2 - 4px_n + x_n^2 = 1$ or $x_n^2 - 4px_n + p^2 - 1 = 0$. Regard this as a quadratic equation in x_n.

Since x_n is an integer, the discriminant must be a perfect square, that is, $(4p)^2 - 4(p^2-1) = 4(3p^2+1)$ is a perfect square, from which $3p^2 + 1 = q^2$, for some positive integer q.

This can be written as $3p^2 = (q-1)(q+1)$.

Note that if p divides both $q-1$ and $q+1$, then it divides their difference, hence $p = 2$, impossible. Since p is a prime, it follows that p^2 divides either $q-1$ or $q+1$. In either case, we have that $p^2 \leq q+1$.

But then

$$3p^2 = (q-1)(q+1) \geq (p^2-2)p^2 \implies 3 \geq p^2 - 2 \implies p = 2, \text{ impossible.}$$

In conclusion, for all $n \geq 2$, x_n is necessarily composite.