April 2016 Problem of the Month

Tangents

Given that

$$\prod_{k=1}^{45}(1+\tan k^\circ) = 2^n,$$

find (with a proof) the value of n. (k° indicates that the angle k is measured in degrees.)

Solution

First note that

$$\begin{aligned} 1+\tan k^\circ &= 1+\frac{\sin k^\circ}{\cos k^\circ} = \frac{\cos k^\circ + \sin k^\circ}{\cos k^\circ} \\ &= \frac{\sqrt{2}\sin(45+k^\circ)}{\cos k^\circ} = \frac{\sqrt{2}\cos(45-k^\circ)}{\cos k^\circ}. \end{aligned}$$

Thus

$$(1+\tan k^\circ)(1+\tan(45-k^\circ)) = \frac{\sqrt{2}\cos(45-k^\circ)}{\cos k^\circ}\cdot\frac{\sqrt{2}\cos k^\circ}{\cos(45-k^\circ)} = 2.$$

It follows that

$$\begin{aligned} \prod_{k=1}^{45}(1+\tan k^\circ) &= (1+\tan 45^\circ)\prod_{k=1}^{22}(1+\tan k^\circ)(1+\tan(45-k^\circ)) \\ &= 2\prod_{k=1}^{22} 2 = 2^{23}, \end{aligned}$$

so $n = 23$.