February 2016 Problem of the Month

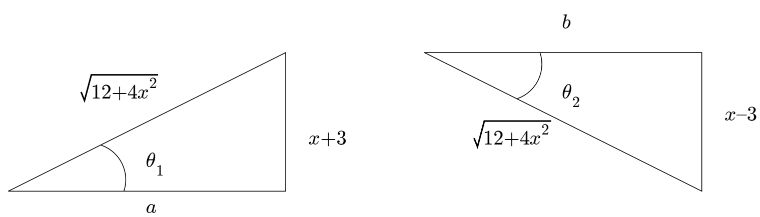
Trigonometry

Show that

$$\sin^{-1}\left(\frac{x+3}{\sqrt{12+4x^2}}\right) - \sin^{-1}\left(\frac{x-3}{\sqrt{12+4x^2}}\right)$$

is constant for $-1 \le x \le 1$, and find the value of the constant.

Solution



Denote the expression as $\theta_1 - \theta_2$. Notice from the figures that the unknown side a of the triangle with angle θ_1 is $\sqrt{12+4x^2-(x+3)^2} = \sqrt{3}(1-x)$, and the unknown side b of the triangle with angle θ_2 is $\sqrt{12+4x^2-(x-3)^2} = \sqrt{3}(1+x)$. Taking the sine of the expression,

$$\begin{aligned}\sin(\theta_1-\theta_2) &= \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 \\ &= \frac{x+3}{\sqrt{12+4x^2}}\cdot\frac{\sqrt{3}(1+x)}{\sqrt{12+4x^2}} - \frac{\sqrt{3}(1-x)}{\sqrt{12+4x^2}}\cdot\frac{x-3}{\sqrt{12+4x^2}} \\ &= \frac{\sqrt{3}}{2}.\end{aligned}$$

Hence the expression equals either $\pi/3$ or $2\pi/3$, but the expression is continuous on $[-1,1]$, so it cannot take on both values. When $x=0$,

$$\sin^{-1}\left(\frac{x+3}{\sqrt{12+4x^2}}\right) - \sin^{-1}\left(\frac{x-3}{\sqrt{12+4x^2}}\right) = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

Thus, the expression must equal $2\pi/3$ on the entire interval.