March 2016 Problem of the Month

Progressions

Let \( a, b, B, c, C, \) and \( d \) be six positive real numbers such that \( a, b, c, d \) is an arithmetic progression and \( a, B, C, d \) is a geometric progression. Find (with a proof) all possible values of \( bc/BC \).

Solution

Let \( a + k = b, b + k = c, \) and \( c + k = d \). Then \( k = b - a \) can be substituted into the second and third of these equations. Solving for \( b \) and \( c \) yields

\[
 b = \frac{2a + d}{3} \quad \text{and} \quad c = \frac{a + 2d}{3}.
\]

Similarly, let \( ar = B, Br = C, \) and \( Cr = d \). Then \( r = B/a \) can be substituted into the second and third of these equations. Solving for \( B \) and \( C \) yields

\[
 B = \sqrt[3]{a^2d} \quad \text{and} \quad C = \sqrt[3]{ad^2}.
\]

Thus

\[
 \frac{bc}{BC} = \frac{\frac{2a + d}{3} \cdot \frac{a + 2d}{3}}{\sqrt[3]{a^2d} \sqrt[3]{ad^2}} = \frac{(2a + d)(a + 2d)}{9ad} = \frac{5}{9} + \frac{2}{9} \left( \frac{a}{d} + \frac{d}{a} \right).
\]

Because \( a \) and \( d \) range over all positive real numbers, so does \( x = a/d \). It is a standard problem in calculus to show that the expression \( x + 1/x \) has a minimum value of 2 and takes on all possible values in the interval \([2, \infty)\). Therefore, the minimum value of \( bc/BC \) is \( 5/9 + 2/9 \cdot 2 = 1 \), and it takes on all possible values in the interval \([1, \infty)\).