March 2016 Problem of the Month

Progressions

Let a, b, B, c, C, and d be six positive real numbers such that a, b, c, d is an arithmetic progression and a, B, C, d is a geometric progression. Find (with a proof) all possible values of bc/BC.

Solution

Let $a + k = b$, $b + k = c$, and $c + k = d$. Then $k = b - a$ can be substituted into the second and third of these equations. Solving for b and c yields

$$b = \frac{2a + d}{3} \quad \text{and} \quad c = \frac{a + 2d}{3}.$$

Similarly, let $ar = B$, $Br = C$, and $Cr = d$. Then $r = B/a$ can be substituted into the second and third of these equations. Solving for B and C yields

$$B = \sqrt[3]{a^2 d} \quad \text{and} \quad C = \sqrt[3]{a d^2}.$$

Thus

$$\begin{aligned} \frac{bc}{BC} &= \frac{(2a + d)/3 \cdot (a + 2d)/3}{\sqrt[3]{a^2 d}\sqrt[3]{a d^2}} \\ &= \frac{(2a + d)(a + 2d)}{9ad} \\ &= \frac{5}{9} + \frac{2}{9}\left(\frac{a}{d} + \frac{d}{a}\right). \end{aligned}$$

Because a and d range over all positive real numbers, so does $x = a/d$. It is a standard problem in calculus to show that the expression $x + 1/x$ has a minimum value of 2 and takes on all possible values in the interval $[2, \infty)$. Therefore, the minimum value of bc/BC is $5/9 + 2/9 \cdot 2 = 1$, and it takes on all possible values in the interval $[1, \infty)$.