MODELS OF THE LEFT VENTRICLE
(From S.Y. Rabbany 1991)

Stress analysis of the heart wall is complex, due in part to nonlinear and time-dependent myocardial stress-strain relationships. The complexity of the left ventricle has been managed by the introduction of several simplifying assumptions. In spite of these assumptions, most of theoretical analyses seem to have compounded the issue by obtaining what appear to be contradictory results.

In a preliminary investigation of the question whether solid elastic wall models of the ventricle may be suitable when experimentally obtained intramyocardial pressure data are taken into consideration, two simple geometric models of the left ventricle are considered (Rabbany et al., 1984). The left ventricle is treated as a homogeneous and isotropic solid elastic body. The structural analysis is undertaken at the continuum level, thus ignoring the microstructure (i.e., assuming uniform mechanical properties for the myocardium). The time-varying geometry of the left ventricle is modeled as both a thick-walled cylinder of constant height and a thick-walled sphere. These shapes are employed since they provide simple geometric extremes, yet retain lucidity in the essential features of the ventricle. Measured ventricular pressure, volume and wall thickness data were adapted from measurements on human subjects.

The variation of the stress from the inner surface to the outer surface, for the cylindrical model, is examined here. For a ring cut by two planes perpendicular to the axis at a unit distance apart, due to the conditions of symmetry, no shearing stresses exist on the sides of the element mm, n1. Considering this element in the cylindrical model, the tangential stress is acting on the sides mm, and m1, whereas the radial stress is normal to the side mn.

**Figure.** Stresses in the thick-walled cylinder. (A) Thin annulus of thickness dz with z-axis is perpendicular to the plane of the figure. (B) Cylindrical volume element of thickness dz.
Radial stress varies with the radius \( r \) in the amount of \((d\sigma_r/dr)dr\) over a distance \( dr \). As a result the normal radial stress on the side \( m,n \) is:

\[
\sigma_r + \frac{d\sigma_r}{dr} dr
\]  

(4-1)

To obtain the equation of equilibrium for this element, the forces are summed in the direction of the bisector of the angle \( d\phi \):

\[
\sigma_r r d\phi + \sigma_t r d\phi - (\sigma_r + \frac{d\sigma_r}{dr} dr) (r + dr)d\phi = 0
\]  

(4-2)

For small angles the sine of the angle is approximated by the angle in radians. Neglecting the quantities of high order terms, the expression relating \( \sigma_t \) to \( \sigma_r \) is as follows:

\[
\sigma_t - \sigma_r - r \frac{d\sigma_r}{dr} = 0
\]  

(4-3)

To obtain a second relation the deformation of the cylinder is viewed as symmetrical with respect to axis and consisting of a radial displacement of all points in the wall of the cylinder. This displacement is constant in the circumferential direction but varies with distance along the radius. With \( u \) denoting the radial displacement at the surface of radius \( r \), the radial displacement at the surface of radius \( r+dr \) is:

\[
u + \frac{du}{dr} dr
\]  

(4-4)

Subsequently, the element \( mnm,n \), undergoes a total elongation in a radial direction of \((du/dr)dr\), or a unit elongation of:

\[
\varepsilon_r = \frac{du}{dr} \frac{dr}{dr} = \frac{du}{dr}
\]  

(4-5)

In the tangential direction the unit elongation of the same element is:

\[
\varepsilon_t = \frac{u}{r}
\]  

(4-6)

where \( u \) denotes radial deformation, and \( r \) the initial radius.

The following set of expressions for the stresses in terms of strains is obtained by employing the generalized Hooke's law for a linearly elastic membrane:
\[
\sigma_r = \frac{E}{(1-\nu^2)} \left( \frac{du}{dr} + \frac{\nu u}{r} \right) \tag{4-7}
\]

\[
\sigma_t = \frac{E}{(1-\nu^2)} \left( \frac{u}{r} + \frac{\nu du}{dr} \right) \tag{4-8}
\]

where \(\sigma_r, \sigma_t\) are the orthogonal stresses in the radial, and tangential directions, respectively; \(E\) is Young's modulus, and \(\nu\) is the Poisson's ratio. These stresses are interdependent since they are expressed in terms of deformation \((u)\). Substituting the values for \(\sigma_r\) and \(\sigma_t\) from Equations (4-7) and (4-8) into the equation of Equilibrium (4-3), the following differential equation is obtained:

\[
\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0 \tag{4-9}
\]

The general solution of this equation is:

\[
u = Ar + \frac{B}{r} \tag{4-10}
\]

Substituting Equation (4-10) its differentiated form:

\[
\frac{du}{dr} = A - \frac{B}{r^2} \tag{4-11}
\]

into Equations (4-7) and (4-8) results in the following two expressions for the radial and tangential stresses:

\[
\sigma_r = \frac{E}{1-\nu^2} \left[ A(1 + \nu) - \frac{B}{r^2} \right] \tag{4-12}
\]

\[
\sigma_t = \frac{E}{1-\nu^2} \left[ A(1 + \nu) + \frac{B}{r^2} \right] \tag{4-13}
\]

The constants \(A\) and \(B\) can be determined from the conditions at the inner and outer surfaces.
of the cylinder where the pressure, (i.e., the radial stress; \(\sigma_r\)) is known. For \(P_i\) denoting the internal pressure and \(P_e\) the external pressure, the conditions at the inner and outer surfaces of the cylinder are:

\[
\sigma_{r_{i/o}} = -P_i \quad ; \quad \sigma_{r_{o/o}} = -P_e
\]  

(4-14)

Substituting the expression for \(\sigma_r\) from Equation (4-12) into Equation (4-14) allows determination of the constants A and B:

\[
A = \frac{(1 - \nu)}{E} \frac{r_i^2 P_i - r_e^2 P_e}{r_e^2 - r_i^2}
\]  

(4-15)

\[
B = \frac{(1 + \nu)}{E} \frac{r_i^2 r_e^2 (P_i - P_e)}{r_e^2 - r_i^2}
\]  

(4-16)

Placement of Equations (4-15) and (4-16) into Equations (4-12) and (4-13) gives the general expressions for these stresses. For the condition of no change in the length of the cylinder \((\varepsilon_l = 0)\) and from the generalized Hooke's law the expression for the longitudinal stresses follows:

Therefore, the expressions for the three orthogonal stresses in our cylindrical model are derived, and follow the classical Lame solution (1852):

Radial stress:

\[
\sigma_l = \nu(\sigma_r + \sigma_t)
\]  

(4-17)

Tangential stress:

\[
\sigma_r = \frac{P_i r_i^2 - P_e r_e^2}{r_e^2 - r_i^2} - \frac{r_i^2 r_e^2 (P_i - P_e)}{r_e^2 (r_e^2 - r_i^2)}
\]  

(4-18)

\[
\sigma_t = \frac{P_i r_i^2 - P_e r_e^2}{r_e^2 - r_i^2} + \frac{r_i^2 r_e^2 (P_i - P_e)}{r_e^2 (r_e^2 - r_i^2)}
\]  

(4-19)
Axial stress:
\[
\sigma_t = \frac{P_i r_1^2 - P_e r_e^2}{r_e^2 - r_1^2}
\]

where \( r_1 \): internal radius; \( r_e \): external radius; \( r \): radius at some point in the wall; \( P_i \): internal (cavity) pressure; and \( P_e \): external pressure.

It is interesting to observe from the above expressions that for the case of linear symmetrical shell the stresses are independent of the material constants. Furthermore, these equations illustrate that for the case of \( P_e = 0 \) the radial stress is always compressive, whereas the tangential and longitudinal stresses are tensile.

Intramycocardial pressure, denoted here by \( P_{im} \), is interpreted as the pressure in a minute fluid pocket in the myocardium. Classically \( P_{im} \) is defined as the negative of the average of three time-varying orthogonal stresses (Sommerfield, 1947):
\[
P_{im} = -\frac{\sigma_r + \sigma_t + \sigma_l}{3}
\]

Upon substitution the following expressions is obtained:
\[
P_{im} = -\frac{P_i r_1^2 - P_e r_e^2}{r_e^2 - r_1^2}
\]

which allow the calculation of intramyocardial pressure as a function of time in a thick-walled cylindrical model of constant height and in a spherical model of the left ventricle respectively. In the particular case when external pressure is zero (\( P_e = 0 \)), equation (4-22) reduce further:
\[
P_{im} = -\frac{r_1^2}{r_e^2 - r_1^2} P_i
\]

In the first generation models, the description of the ventricle was reduced to its basic aspects. The wall material was taken to have linear properties, while geometric nonlinearity was ignored. These approximations may well imply that the calculated \( P_{im} \) is not quantitatively correct. Nevertheless, both models expose fundamental discrepancy with measured intramyocardial pressure values. Computed values are negative, while experimental values are consistently positive, independent of the measurement method employed.