On the Dressing of String Solutions

Richard M. Myers
Hofstra University
Departments of Physics and Mathematics

Abstract

The differential systems known as integrable systems represent one of the few classes of partial differential systems for which there is a systematic approach. Interest in these systems first arose from the study of Hamiltonian systems in physics, particularly since few non-integrable systems were found to admit closed form solutions. Since then, integrable systems continue to be a prime subject of study since they appear now in the field theories which describe modern physical theories, including string theory which is the source of the AdS_5 minimal surface problem presented in this paper. The classical theory of integrable systems is presented to motivate and develop the theory of dressing transformations, we then discuss the application of these transformations to the minimal surface problem.

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§1 Introduction

The classical theory of integrable systems describes a class of partial differential equations which represent one of the few general classes for which explicit solutions can be found. Liouville proved that these systems are solvable via a single integration and a series of algebraic function inversions, though the key insight was the use of conserved quantities to change variables such that the system admits a trivial solution.

Since Liouville, the role of conserved quantities in theoretical physics has only increased in importance, particularly with developments such as Emmy Noether's theorem near the turn of the 20^{th} century which relates conserved quantities to symmetries. In the study of integrable systems, however, this focus on conserved quantities lead to the development of what are known as Lax pairs which, if they can be found, are able to convert a Hamiltonian system in to a matrix equation, and one which immediately yields all conserved quantities.

However, some Lax pairs do not correctly generate all of the conserved quantities unless a complex parameter, the spectral parameter, is introduced to the pair's definition. It is the study of the possible complex pole structures of spectral parameter which gave rise to the Zakharov-Shabat construction. This construction yields a beautiful result about integrable systems: the pole structure completely determines the physical model that the Lax pair describes. All other information is either incidental or a product of particular additional constraints placed on the model which are not inherent.

This understanding of the Lax formalism is imperative when moving to integrable field theories and discussing the dressing transformations. These are transformations which act on a known solution of an integrable system to generate a new solution in a non-trivial manner. The entire theory of dressing transformations, however, is reliant on leaving the pole structure invariant, and this producing a new solution which is within the same physical model.

Once we have discussed the dressing transformations and presented two examples of their application, we can begin to discuss integrable system which results from the string theoretic problem of finding new minimal surfaces in AdS_5 space. This is the system whose dressing we seek to develop and discuss.

§2 Integrable Systems

The study of classical dynamical systems is, mathematically speaking, no more than the study of a specific subclass of differential equations which happen to be of particular interest due to their interpreted description of physical phenomena. These systems generically exist in a $2n$ dimensional space referred to as the phase space with coordinates (p_i, q_i). In this space there exists also a function $H(p_i, q_i)$, the Hamiltonian, which is of particular interest

since the solutions to the dynamical system are paths in the phase space which satisfy a differential system defined by this function H.

We can show that on a path satisfying this differential system, the Hamiltonian is conserved, so that any solution to the system must lie on the level manifold defined by $H = E$ for some constant E. This statement, however, is not generally enough to specify a path through the phase space. We therefore begin in this chapter with a review of Liouville integrability, and stress the importance of conserved quantities in defining paths through the phase space.

With this framework in hand, we will introduce the notion of Lax pairs, a pair of matrices which have the ability to generate not only the Hamiltonian system, but also all conserved quantities. The power and versatility of the Lax matrices, however, does not end with the ability to write the governing equations of a dynamical system in matrix form, nor with the ability to generate the conserved quantities. There exist some systems which can be put into the Lax form, but cannot generate all conserved quantities. This forces us to introduce the so called spectral parameter into the Lax matrices, rectifying the issues with generating the conserved quantities.

It is this spectral parameter that will lead us to a characterization of the Lax matrices and a mathematical framework which is usually taken to be the starting point for any work done with dressing solutions or solitons. Although mathematical results of this first chapter are not strictly necessary to a development of the dressing technique, the concepts explored throughout this chapter provide the underlying basis for the dressing technique and why it works.

The development here will follow that of Olivier Babelon in [1], though some material will be omitted here as it is not necessary to our objectives. Further, the development here endeavors more strongly to highlight the parallels in form when moving to the dressing technique in §3.

§2.1 Liouville Integrable Systems

To begin, we will suppose that our systems of interest lie in a space, M, of dimension $2n$ which is covered by the n position coordinates q_i and the n momentum coordinates p_i. The Hamiltonian is a function in this so called phase space, which we will denote by $H(p_i, q_i)$. We are then interested in paths through the phase space which satisfy the system

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2.1.1}$$

where the dot refers to a derivative with respect to the variable parametrizing the path through phase space, usually taken to be time.

For any functions on the phase space, say F and G, we may define the Poisson bracket $\{F, G\}$ by

$$\{F, G\} = \sum_i \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}, \tag{2.1.2}$$

from which it immediately follows that, along the path satisfying (2.1.1),

$$\dot{F} = \{H, F\}. \tag{2.1.3}$$

In particular, we note that $\{H, H\} = 0$, so the Hamiltonian itself is conserved along the path through phase space. This implies that any path satisfying (2.1.1) must lie on the manifold defined by $H = E$ for some constant E.

Now, while systems of this mathematical form are of interest in physics, there are very few such systems which may actually be solved exactly. There is, however, a subclass of these which Liouville proved may be solved by computing a finite number of integrals and solving a finite number of algebraic equations, the procedure is known as quadrature.

Towards this end, we must first examine the connection between the Poisson bracket and 2-forms on the phase space. To see the connection, consider the operator

$$\{*, *\} = \sum_i \frac{\partial}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i}\frac{\partial}{\partial p_i}, \tag{2.1.4}$$

which is a completely antisymmetric outter product of tangent vectors on the phase space. Since every vector space is isomorphic to its dual, this operator corresponds bijectively to the 2-form

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i. \tag{2.1.5}$$

Since

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \tag{2.1.6}$$

it follows immediately that the $\{*, *\}$ operator is never degenerate, and so neither is[1] ω. In this way, a non-degenerate Poisson bracket corresponds bijectively to a non-degenerate 2-form in the dual defined on M and further, $d\omega = 0$. Such a form is referred to as the symplectic form on M and reads as (2.1.5) in canonical coordinates.

So, we may now define what we mean by an integrable system.

Definition 1. A Hamiltonian system is Liouville integrable if there exist n independent functions F_i, our conserved quantities, such that $\{H, F_i\} = 0$ and $\{F_i, F_j\} = 0$, this second condition known also as convolution.

[1]Note that ω may be degenerate when restricted to a subspace of M, and in fact, this will be relied upon in proving that Liouville integrable systems are solvable by quadrature.

The requirement of independence ensures that at any point on M, the $\mathrm{d}F_i$ are linearly independent, and so the level manifold defined by $F_i = f_i$ for some constants f_i has dimension n at all points. Furthermore, we cannot have more than n conserved quantities. If there were more than n, then considering the dual of the convolution condition, we would observe more than n linearly independent 1-forms wedge to zero. However, (2.1.6) implies that this is impossible in the canonical basis, and so is impossible in the $\mathrm{d}F_i$ basis. Thus, we know that the dimension must be at most n, and we require it to be exactly n to proceed with quadrature. This is perhaps seen more readily if we think of ω as the antisymmetric analog of the metric tensor which may be more familiar to the reader.

So then, what is quadrature?

The Liouville Theorem 1. The solution of the equations of motion of a Liouville integrable system is obtained by quadrature.

Proof. Let $\alpha = \sum_i p_i \mathrm{d}q_i$ be the canonical 1-form and $\omega = \mathrm{d}\alpha$ be the symplectic 2-form on the phase space M. The goal here is to construct the canonical transformation $(p_i, q_i) \to (F_i, \psi_i)$ such that the conserved quantities F_i replace the canonical momenta as coordinates, so

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i = \sum_i \mathrm{d}F_i \wedge \mathrm{d}\psi_i. \tag{2.1.7}$$

If we can do this, then the equations of motion reduce to

$$\begin{aligned} \dot{F}_i &= \{H, F_i\} = 0, \\ \dot{\psi}_i &= \{H, \psi_i\} = \frac{\partial H}{\partial F_i} = \Omega_i, \end{aligned} \tag{2.1.8}$$

where the Ω_i depend only on the F_i, and so are constants in time. The dependence of the Ω_i on the F_i alone would follow immediately if we could only show that H does not depend on the ψ. But if we recall that the F_i were chosen such that $\{H, F_i\} = 0$, we need only observe that this implies

$$0 = \sum_j \frac{\partial H}{\partial F_j}\frac{\partial F_i}{\partial \psi_j} - \frac{\partial H}{\partial \psi_j}\frac{\partial F_i}{\partial F_j} = -\sum_j \frac{\partial H}{\partial \psi_j}\delta_{ij} = -\frac{\partial H}{\partial \psi_i}, \tag{2.1.9}$$

where we have used the fact that $\frac{\partial F_i}{\partial F_j} = \delta_{ij}$ and $\frac{\partial F_i}{\partial \psi_j} = 0$, which follows from the statement that the ψ_i and F_i form an *intrinsic* coordinate basis for the manifold M. So, the Hamiltonian cannot depend on the ψ_i, and therefore neither can the $\Omega_i = \frac{\partial H}{\partial F_i}$.

Once the equations of motion are reduced to this form, the solution becomes trivial:

$$\begin{aligned} F_i(t) &= F_i(0), \\ \psi_i(t) &= \psi_i(0) + t\Omega_i. \end{aligned} \tag{2.1.10}$$

To actually construct the desired transformation, we first let M_f be the level manifold defined by $F_i(p,q) = f_i$ for some constants f_i and solve for the p_i algebraically to write $p_i = p_i(f,q)$. We then define the generating function S by

$$S(F,q) = \int_{m_0}^{m} \alpha = \int_{q_0}^{q} \sum_i p_i(f,q)\mathrm{d}q_i \tag{2.1.11}$$

where the integration path lies in M_f, starting at a reference point $(p(f,q_0),q_0)$ and ending at a generic point $(p(f,q),q)$. A more detailed look at this so-called generating function can be found in [5].

For the moment, let us suppose that this integration is path independent so that S exists and is single-valued. Then since $p_j = \frac{\partial S}{\partial q_j}$, we would like to define the coordinates ψ_j by

$$\psi_j = \frac{\partial S}{\partial F_j}. \tag{2.1.12}$$

It therefore follows that if we consider the differential of S,

$$\mathrm{d}S = \sum_i \frac{\partial S}{\partial F_i}\mathrm{d}F_i + \frac{\partial S}{\partial q_i}\mathrm{d}q_i = \sum_i \psi_i \mathrm{d}F_i + p_j \mathrm{d}q_i. \tag{2.1.13}$$

But now, $d^2S = 0$, thus $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i = \sum_i \mathrm{d}F_i \wedge \mathrm{d}\psi_i$ follows immediately, and so the transformation is the canonical one described above.

The only thing that remains, then, is to show that S is actually a well-defined function. However, if S is zero on any given loop in M_f, then it must be the case that S is also independent of any path in M_f. Therefore, if we define an arbitrary loop $\partial\Sigma$ on M_f with interior Σ, we may write

$$\int_\Sigma \mathrm{d}\alpha = \int_{\partial\Sigma} \alpha, \tag{2.1.14}$$

by Stoke's Theorem. It would therefore be sufficient to prove that $\mathrm{d}\alpha|_{M_f} = \omega|_{M_f} = 0$. To show this, we define X_i to be the Hamiltonian vector field associated with F_i defined by $\mathrm{d}F_i = \omega(X_i,\cdot)$, which may be inverted to find[2]

$$X_i = \sum_j \frac{\partial F_i}{\partial q_k}\frac{\partial}{\partial p_k} - \frac{\partial F_i}{\partial p_k}\frac{\partial}{\partial q_k}. \tag{2.1.15}$$

[2]It is a simple exercise to check that (2.1.15) is indeed the correct inverse.

These vector field must then be tangent to the manifold M_f because $X_i(F_j) = \{F_i, F_j\} = 0$, which follows from the definition of M_f as a level surface and the convolution condition: the gradients of the F_j necessarily do not lie in M_f. Now, since the F_i were assumed to be independent functions, the X_i form a good basis at each point in M_f. But $\omega(X_i, X_j) = \mathrm{d}F_i(X_j) = 0$. Thus, it must be the case that $\omega|_{M_f} = 0$, and so S is well defined. □

To surmise, we begin this procedure by finding n quantities $F_i(p_i, q_i)$ which are conserved in time and which also satisfy the convolution condition $\{F_i, F_j\} = 0$. We then define the generating function $S(F, q)$ given by (2.1.11) which allows us to define the cyclic coordinates ψ_i by (2.1.12). With these definitions in hand, we need only invert the relationships between the two coordinate sets, which is a purely algebraic problem. In this way, we are able to transform to a coordinate system by computing a single line integral, n derivatives of that integral, and a series of algebraic operations to invert the functional relationships between the two coordinate systems.

Of course, even if we are capable of computing the integral, which itself may easily be a non-trivial task, inverting the coordinate relationship may also present a non-trivial problem. For example, if the conserved quantities involve combinations of exponential, hyperbolic, and trigonometric functions, then it is possible for the inversion problem to be a transcendental one.

Though this problem of implementing the canonical transformation is most naturally thought of as a symbolic one, the entire procedure is sufficiently systematic that we could conceivably implement the procedure numerically, though if we are to take this route, we might as well apply our numerical techniques to the original Hamiltonian system. The only instance in which we might want to actually implement this transformation numerically is if we are concerned that the original system is sufficiently chaotic that small numerical errors could result in drastic changes in solution. In a case such as this, moving first through an integration, (2.1.11), could add stability to numerically generated output since numerical integration techniques are generically more stable than those for derivatives.

§2.2 Lax Pairs

We define a Lax pair to be a pair of square matrices L, M which are functions on the phase space of the Hamiltonian system such that the equations of motion (2.1.1) may be written in the form

$$\dot{L} = [M, L], \tag{2.2.1}$$

where $[M, L] = ML - LM$ is the usual commutator. We may further verify that (2.2.1) has solutions of the form

$$L(t) = g(t)L(0)g^{-1}(t) \tag{2.2.2}$$

for some invertible function $g(t)$ defined by

$$M = \frac{\mathrm{d}g}{\mathrm{d}t} g^{-1}. \tag{2.2.3}$$

The form (2.2.2) then implies that the quantities $\mathrm{Tr}(L^k)$ are conserved for each $k \in \mathbb{N}$.

We will now present an example of a Lax pair which will lead us to the next stop in our development. However, there is a bit of a disclaimer that must accompany this example: some of the following procedures may appear arbitrary, and such a judgement would not be far from the truth. So far as the author is aware, there is no general procedure by which, given a system, a Lax pair may be produced, nor a general procedure for introducing the spectral parameter. So, the steps in this example are followed merely because they are the steps which happen to generate the desired result, not because there is an algorithm omitted from this work.

The example is the Euler top, which is a rotating solid body attached to a fixed point. There are no external forces acting on the top and we choose a reference frame which rotates along with the top, the origin placed at the fixed point and the rotational axis such that the inertial tensor, $I_{ij} = \int (\mathbf{x}^2 \delta_{ij} - x_i x_j)\rho(x)\mathrm{d}x$ with $\rho(x)$ the mass density, is diagonalized.

We shall suppose that $\mathbf{J}$ is the angular momentum of the top as seen in the rotating frame. Then $\mathbf{J} = I\omega$ where $I = \mathrm{diag}(I_1, I_2, I_3)$ and ω is the rotation vector of the moving frame. Then

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = -\omega \wedge \mathbf{J} = -\omega \times \mathbf{J} \tag{2.2.4}$$

are the equations of motion, which simply express the conservation of $\mathbf{J}$ in the absolute frame. But if we define the 3×3 matrices $J_{ij} = \epsilon_{ijk} J_k$ and $\Omega_{ij} = \epsilon_{ijk}\omega_k$, where ϵ_{ijk} is the completely antisymmetric tensor, then (2.2.4) is equivalent to

$$\frac{\mathrm{d}J}{\mathrm{d}t} = [\Omega, J], \tag{2.2.5}$$

which is now in Lax form. However, if we now check the quantities $\mathrm{Tr}(L^k)$, we see that all quantities are either zero or functions of $\mathbf{J}^2$. The Hamiltonian, despite being conserved, will never appear. Well this is a bit of an issue. However, there is a way to rectify this problem. If we define the matrix $\mathcal{I} = \mathrm{diag}(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ with $\mathcal{I}_k = \frac{1}{2}(I_i + I_j - I_k)$ where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Then $J = \mathcal{I}\Omega + \Omega\mathcal{I}$. Setting

$$L(\lambda) = \mathcal{I}^2 + \frac{1}{\lambda} J, \quad M(\lambda) = \lambda\mathcal{I} + \Omega, \tag{2.2.6}$$

where λ is a free complex parameter, the so called spectral parameter.

Now,

$$\dot{L}(\lambda) - [M(\lambda), L(\lambda)] = [J, \mathcal{I}] + [\mathcal{I}^2, \Omega] + \frac{1}{\lambda}(\dot{J} + [J, \Omega]), \tag{2.2.7}$$

where the first two terms cancel each other and the third term vanishes by definition of J and Ω as a Lax pair. Thus, this pair $L(\lambda), M(\lambda)$ is a Lax pair for all λ. Now, however, both H and $\mathbf{J}^2$ will appear among the conserved quantities.

The most important thing here is to note the dependence on this spectral parameter and that the Lax equations must hold identically for all λ. This allows each trace term $\mathrm{Tr}(L(\lambda))^k$ to generate multiple conserved quantities. Without including the spectral parameter, we have effectively evaluated the system at $\lambda = \infty$, projecting out some of the information about the conserved quantities. So, the pole structure really is a necessary component to describing the model.

§2.3 The Zakharov-Shabat Construction

As we noted above, there does not yet exist an algorithm for constructing a Lax pair which generates a given system. However, there does exist a general procedure for constructing Lax pairs which give rise to integrable systems. This method is the Zakharov-Shabat construction and seeks matrices $L(\lambda)$ and $M(\lambda)$ which depend on the spectral parameter λ such that the Lax equation

$$\partial_t L(\lambda) = [M(\lambda), L(\lambda)] \tag{2.3.1}$$

is equivalent to some integrable system. As we will see, the key to this method lies in the polar structure of the Lax pair. There is, however, an unfortunate quantity of notation that accompanies these efforts. We therefore begin with the following:

We will suppose that $f(\lambda)$ is a matrix-valued rational function, and we will further suppose that it is rational in polynomials of λ. If $f(\lambda)$ has poles of order n_k at the finite[3] points λ_k, when we may decompose $f(\lambda)$ by

$$f(\lambda) = f_0 + \sum_k f_k(\lambda), \quad \text{with} \quad f_k(\lambda) = \sum_{r=-n_k}^{-1} f_{kr}(\lambda - \lambda_k)^r, \tag{2.3.2}$$

for some constants f_0, f_{kr}. We make the observation that this sort of decomposition is precisely the same one used to integrate by partial fraction, which is equivalent to an application of the residue theorem to a rational function.

[3] A rational function with no pole at infinity must only attain a constant value at infinity, and therefore will have no powers higher than zero in the power series expansion. Thus, all such rational functions have finite Laurent expansions.

If we refer to the $f_k(\lambda)$ as the polar part about λ_k, then about λ_k, we may write

$$f(\lambda) = f(\lambda)_+ + f(\lambda)_- \tag{2.3.3}$$

where $f(\lambda)_+$ is regular about the pole and $f(\lambda)_- = f_k(\lambda)$ is singular about the pole.

Now, we will consider the matrices $L(\lambda)$ and $M(\lambda)$ with dimension $N \times N$ which we assume to be rational functions of the spectral parameter. Letting $\{\lambda_k\}$ be the collection of both of their poles[4], that is, the union of the two collections of poles, we may write these two matrix functions in the notation presented above:

$$L(\lambda) = L_0 + \sum_k L_k(\lambda), \quad \text{with} \quad L_k(\lambda) = \sum_{r=-n_k}^{-1} L_{kr}(\lambda - \lambda_k)^r, \tag{2.3.4}$$

and

$$M(\lambda) = M_0 + \sum_k M_k(\lambda), \quad \text{with} \quad M_k(\lambda) = \sum_{r=-m_k}^{-1} M_{kr}(\lambda - \lambda_k)^r, \tag{2.3.5}$$

where the n_k are the pole orders of $L(\lambda)$ at λ_k, and similarly m_k are the pole orders of $M(\lambda)$ at the respective poles. Note, however, that we indexed the collection of all poles, so there may be some locations which are poles of one but not the other function. In these cases, we observe that the sum in the polar part runs from zero to -1, and therefore is zero. Or, if it would seem more appropriate, simply set the necessary coefficients to zero. The result is the same in either case.

It is here that we reach an important conceptual milestone. If we plug (2.3.4) and (2.3.5) into the Lax equation (2.3.1), it must be the case that both the left and right hand side have poles of the same order at the same locations since the Lax equation is required to hold identically for all λ.

If we consider the pole at the location λ_k, the left hand side of the Lax equation has a pole of order n_k, however, the right hand side potentially has a pole of order up to $n_k + m_k$. Thus, we really have two types of equation. The first we find by forcing the right hand side to have a pole of order n_k at λ_k. To do this, we must set to zero the m_k coefficients which multiply pole terms of order higher than n_k. We observe that these equations do not involve time derivatives, and so cannot be the dynamical equations, but rather are simply constraints placed on the form of M_k. We shall see later that this form of constraint has a physical interpretation and is not only a formal mathematical constraint on the allowable form of the M_k matrices.

[4]We shall also assume that these poles themselves do not depend on time. It is possible to allow the pole locations to depend on time, and it is the author's understanding that this technique is utilized in generating interesting black hole geometries such as the torus, however allowing the poles to move in time requires additional care which is not necessary to this work.

The second type of equation comes from demanding that the coefficients of pole terms of the same order on both the right and left hand sides must be equal. These equations do indeed contain time derivatives, and are therefore the true dynamical equations of the system. The key here is that, once we specify the pole structure, that is, the pole locations and orders, we have, in a very real sense, almost completely defined the dynamical system up to a redefinition of variables.

Since this is such an important conceptual point, we must be quite careful to ensure that this idea is conveyed precisely. By defining the pole structure, we have defined the number of variables and the number of equations present in the system. Conceptually, his means that if we wish to study an integrable system S with at most as many governing equations as we have dynamical pole equations, then there exists a function, say f, which maps the coefficients of L to the variables in S and a collection of matrices M_k which mix the variables of S together in the correct way when computing the commutator. This sort of idea of course allows for the possibility that f is not a bijective function, and may well map some of the coefficients to zero. But this gives us a sense that we have defined something analogous to the permutation group in finite group theory. In finite group theory, if we are given a group, then we know that there is some permutation group which can be mapped homomorphically down to the group we started with. This is the idea we are trying to display here, with f playing a role analogous to the group homomorphism. To surmise: deciding the pole structure is equivalent to deciding the model the Lax pair describes.

So, the final task for this section is to better characterize the M_k matrices. To do so, we begin with a Proposition:

Proposition 1. Assuming that $L(\lambda)$ has distinct eigenvalues in a neighborhood of λ_k, one can perform a regular similarity transformation[5] $g^{(k)}(\lambda)$ diagonalizing $L(\lambda)$ in a vicinity of λ_k. That is,

$$L(\lambda) = g^{(k)}(\lambda)A^{(k)}(\lambda)g^{(k)-1}(\lambda), \tag{2.3.6}$$

locally, where $A^{(k)}(\lambda)$ is some diagonal matrix with a pole of order n_k at λ_k. As a result, we may give the global statement

$$\begin{aligned} L &= L_0 + \sum_k L_k, \quad \text{with} \quad L_k = \left(g^{(k)}(\lambda)A^{(k)}(\lambda)g^{(k)-1}(\lambda)\right)_-, \\ M &= L_0 + \sum_k M_k, \quad \text{with} \quad M_k = \left(g^{(k)}(\lambda)B^{(k)}(\lambda)g^{(k)-1}(\lambda)\right)_-, \end{aligned} \tag{2.3.7}$$

where $B^{(k)}(\lambda)$ has a pole of order m_k at λ_k. Moreover, the Lax equation implies that $B^{(k)}(\lambda)$ is diagonal.

[5]We place the index k in the superscript to avoid conflict with our notational convention that a subscript refers to the pole term about λ_k.

Proof. If the point λ_k is a pole of $L(\lambda)$, then when we say that we demand the eigenvalues of $L(\lambda)$ to be distinct in a neighborhood of λ_k, what we mean is that the matrix $Q(\lambda) = (\lambda - \lambda_k)^{n_k} L(\lambda)$, which is regular at λ_k, must have distinct eigenvalues, and so can be diagonalized by a regular matrix $g^{(k)}(\lambda)$. So, (2.3.6) clearly holds for some matrix $A^{(k)}(\lambda)$.

If we now define $B^{(k)}(\lambda)$ by

$$M(\lambda) = g^{(k)}(\lambda) B^{(k)}(\lambda) g^{(k)-1}(\lambda) + \partial_t g^{(k)}(\lambda) g^{(k)-1}(\lambda) \tag{2.3.8}$$

then we may take a derivative of (2.3.6) to find

$$\begin{aligned} \dot{L} =& \partial_t g A g^{-1} + g \dot{A} g^{-1} + g A \partial_t g^{-1} = g \left[\dot{A} + g^{-1} \partial_t g A - A g^{-1} \partial_t g \right] g^{-1}, \\ =& g \left(\dot{A} + [g^{-1} \partial_t g, A] \right) g^{-1} \end{aligned} \tag{2.3.9}$$

where we have suppressed the index (k) and have applied the identity[6] $\partial g = -g \partial g^{-1} g$. The right hand side of the Lax equation then reads

$$[M, L] = [gBg^{-1}, gBg^{-1}] + [\partial_t g g^{-1}, gAg^{-1}] = g[B, A] g^{-1} + g[g^{-1} \partial_t g, A] g^{-1}, \tag{2.3.10}$$

where we have again suppressed the index k. But now setting (2.3.9) equal to (2.3.10) to form the Lax equation, we find that the gauge term $\partial_t g g^{-1}$ cancels and we are left with

$$\dot{A}^{(k)}(\lambda) = [B^{(k)}(\lambda), A^{(k)}(\lambda)]. \tag{2.3.11}$$

If $\mathbf{e}_j$ is the canonical basis in the new system we have transformed to with the similarity transformation $g^{(k)}(\lambda)$, then we may consider the matrix elements of (2.3.11), dotting on the left and right by $|\mathbf{e}_j\rangle$. In particular, by inserting a complete set of states,

$$\langle \mathbf{e}_j | [B, A] | \mathbf{e}_j \rangle = \sum_i \langle \mathbf{e}_j | B | \mathbf{e}_i \rangle \langle \mathbf{e}_i | A | \mathbf{e}_j \rangle - \langle \mathbf{e}_j | A | \mathbf{e}_i \rangle \langle \mathbf{e}_i | B | \mathbf{e}_j \rangle, \tag{2.3.12}$$

but since A is diagonal in this basis, $\langle \mathbf{e}_j | A | \mathbf{e}_i \rangle = \delta_{ji}$, killing the summation and forcing $i = j$. But then the remaining single term in (2.3.12) cancels with its other half in the commutator, resulting in zero. It then follows that the elements of A in (2.3.11) are conserved, as they should be since the traces of the powers of L are our conserved quantities. Moreover, an analogous argument formed by considering the off-diagonal entries of the commutator implies that if the entries of A are distinct, then B must also be diagonal.

[6] This identity follows from the observation that $\partial(g^{-1} g) = 0$ since $g^{-1} g = 1$ has no functional dependencies, along with an application of the product rule, taking care to preserve the order of multiplication.

Finally, since the term $\partial_t g^{(k)} g^{(k)-1}$ is regular, it does not contribute to the singular part M_k of M at λ_k. Therefore, we may neglect whatever regular part might be picked up from $B^{(k)}$ to write $M_k = (g^{(k)} B^{(k)} g^{(k)-1})_-$, which depends only on $B_-^{(k)}$. All together, this proves the Proposition. □

With this, we have clarified the necessary structure of the Lax matrix. Only the singular parts of the matrices $A^{(k)}$ and $B^{(k)}$ actually contribute to the L_k and M_k matrices. The singular matrices $A_-^{(k)}$ must have the form

$$A_-^{(k)} = \sum_{r=-n_k}^{-1} A_{k,j} (\lambda - \lambda_k)^r. \tag{2.3.13}$$

It then follows that the only free parameters in the full Lax matrix $L(\lambda)$ are those in L_0, the diagonal matrices $A_{k,j}$, and jets of regular matrices $\hat{g}^{(k)}$ of order $n_k - 1$,

$$\hat{g}^{(k)} = \sum_{r=0}^{n_k - 1} g_{k,j} (\lambda - \lambda_k)^r, \tag{2.3.14}$$

which are defined up to right multiplication by a regular diagonal matrix $d^{(k)}(\lambda)$. We restrict the jets to be of order $n_k - 1$ because when we write the Lax matrix as in (2.3.7) so

$$L_k = (\hat{g}^{(k)} A_-^{(k)} \hat{g}^{(k)-1})_-, \tag{2.3.15}$$

any free parameters introduced by powers higher $n_k - 1$ in the jets are removed by taking the singular part of the product. So, we may as well define them away to begin with. We will further point out that these elements of these jets are indeed sources of free parameters since we may well have a different jet $\hat{g}^{(k)}$ at each pole, so these matrices represent a local gauge rather than a global one, and hence cannot simply be removed by a change in coordinates. They may only be removed about a single pole at a time. So, at best, unless some of the $\hat{g}^{(k)}$ are equal, if there are N pole locations, then $N - 1$ of these jets cannot be rendered trivial.

These jets, however, are not generally functions of the conserved variables, whereas since diagonal matrices commute, (2.3.11) implies that the diagonal of $A^{(k)}$ is conserved about its respective pole. Of course, this construction follows only from the Lax equation. To ensure that a Lax system is also integrable, additional constraints must be imposed. One such constraint is that the conserved quantities and their associated position variables must form a good basis, so the jets must inherit time dependence only from the position variables. A characterization of the integrability constraints will not be necessary to our development here and so we shall not discuss it beyond this point. See [1] for explorations into this discussion.

To this point then, we have characterized the necessary structure of the Lax matrix L, and therefore also the allowable sources of dynamical variables given a particular choice in pole structure. We can then also ask what might be said of the M matrix. It turns out that no additional degrees of freedom are obtained from this matrix, as shown by the following Proposition:

Proposition 2. Let $L(\lambda)$ be a Lax matrix of the form (2.3.4). The general form of the $M(\lambda)$ matrix such that the orders of the poles match on both sides of the Lax equation is $M = M_0 + \sum_k M_k$ where

$$M_k = \left(P^{(k)}(L,\lambda)\right)_- \tag{2.3.16}$$

where $P^{(k)}(L,\lambda)$ is some polynomial in $L(\lambda)$ whose coefficients are rational in λ and $(\ \ )_-$ denotes the singular part at $\lambda = \lambda_k$.

Proof. To show that this is actually a solution, we need only check that, about λ_k, $[M_k, L]$ has order at most as large[7] as that of L since $[M_0, L]$ will trivially have the correct order.

But since for any polar function $f(\lambda)$, we know that $f(\lambda) = f(\lambda)_- + f(\lambda)_+$, it follows that

$$\begin{aligned}[M_k, L] &= \left[\left(P^{(k)}(L,\lambda)\right)_-, L\right] = \left[P^{(k)}(L,\lambda) - \left(P^{(k)}(L,\lambda)\right)_+, L\right] \\ &= -\left[\left(P^{(k)}(L,\lambda)\right)_+, L\right],\end{aligned} \tag{2.3.17}$$

where we have used the fact that any polynomial in L will commute with L. This clearly has order at most n_k, so (2.3.16) is sufficient to satisfy the pole-order constraint.

To show that this form is also necessary, we return to the gauge in which we diagonalize by Proposition 1 about λ_k. Then since $A^{(k)}$ is a diagonal $N \times N$ matrix whose entries are assumed to be distinct and nonzero about λ_k, the powers 0 to $N-1$ of $A^{(k)}$ form a basis in the space of diagonal matrices of this size[8]. We may therefore write

$$B^{(k)} = P^{(k)}(A^{(k)}, \lambda) \tag{2.3.18}$$

for some polynomial $P^{(k)}(A^{(k)}, \lambda)$ with degree at most $N-1$ in $A^{(k)}$. If we then revert to the non-diagonal gauge by inserting (2.3.18) into (2.3.7), we find that $M_k = \left(P^{(k)}(L,\lambda)\right)_-$. $\square$

[7] We do not need to require equality since if the order is $j < n_k$, then the dynamical equations that correspond to the orders between j and n_k represent the conservation those conjugate momenta since, at those orders, the Lax equation simply reads $\dot{L} = 0$.

[8] For a detailed explanation of this fact, see Appendix 1.

Remark. One implication of this Proposition is that the pole structure of M and L must be the same, and if they are not, we are free to force them to be. To see this, suppose that λ_k is a pole of $M(\lambda)$ but not of $L(\lambda)$. Then redefining $M(\lambda) \to M(\lambda) - P^{(k)}(L, \lambda)$ leaves the Lax equation unchanged since the polynomial in L will vanish in the commutator. However, this new $M(\lambda)$ will be regular at λ_k since the singular part of $P^{(k)}(L, \lambda)$ about λ_k will cancel the pole present there in the original $M(\lambda)$.

So, the M matrix must be a function of the matrix elements of the Lax matrix, and hence a function purely of the dynamical variables. It then follows that when we discussed the constraints imposed on the M matrix by the pole structure, were actually discussing non-dynamical constraints on the dynamical variables, such as constraining a particle to be bound to, say, the surface of a cylinder.

This is, again, a conceptual milestone in our development. Till now we have seen that specifying the pole structure specifies the model in which we are working. The elements of the L matrix are our dynamical variables, and diagonalizing L at each pole gives the conserved quantities of the system. By imposing the pole-order condition, we also found that there may be m_k non-dynamical constraint equations about each pole, Proposition 2 implies that these constrains are really algebraic constraints on the dynamical variables rather than dynamical ones. Proposition 2 also shows that we are only free to choose a polynomial in the dynamical variables to satisfy these constraints, to specify the dynamical flows.

Intuitively, we can imagine these results in the following manner: The pole structure is specified a priori. This then allows us to construct the L matrix as in (2.3.4). Then the elements of the L_{kr} matrices are the dynamical variables. For the moment, we shall suppose that these are all distinct[9]. If we then choose the identity or zero polynomial for M_k about each pole, then we have declared $\dot{L} = 0$, so by setting each of the coefficients in the Laurent expansion of $\dot{L}$ to zero, we find that all of our dynamical variables are conserved.

If we instead pick some non-trivial polynomial about one of the poles, then the commutator in the Lax equation may no longer be zero. This would then cause the dynamical variables in the polynomial defining M_k to mix themselves in with the those in L. The dynamical equations are no longer trivial because we have now introduced constraints into the system in the same way that the dynamical equations for a free particle in three dimensional space become non-trivial if we declare that the particle must now also live on the surface of a cylinder. The model has not changed since we are still dealing with a free particle, all that has changed is the introduction of purely algebraic constraints on the dynamical variables such that the equations of motion are no longer trivial. This is precisely what the results of

[9]If we like, we could allow some of these entries to be functions of the others, however, we could simply cast this statement in terms of a purely algebraic constraint which, as we will discuss, amounts to no more than some choice in polynomial to define M_k.

the Zakharov-Shabat construction describe, a notion familiar to our experience with simple classical dynamical systems now cast into a language which applies quite generally, even to systems which may not be Liouville integrable.

§2.4 The Wave Function

In developing the Zakharov-Shabat construction, we found that the matrix $M(\lambda)$ was completely determined by choosing some collection of polynomials in the L matrix, one for each pole present in the system. In the ensuing discussion, we determined that this choice in polynomials amounted to a choice in the algebraic relationship between the dynamical variables, and hence the dynamical flows[10].

In this discussion, care was taken to point out how, though the pole structure, and therefore the model, may remain fixed, we are able to construct a system which is, at least mathematically, a special case of the most general trivial system. Since we are free to choose these polynomials without altering the model, we immediately find that there are, in fact, infinitely many ways to choose the M matrix. In this section we seek to organize and characterize this collection and define the so called integrable hierarchy.

To begin, we define an elementary flow to correspond to one of the diagonal $B_-^{(k)}$ matrices, specifically one which has only a single pole of order n at λ_k and in the diagonal entry α. We will also define a so-called elementary time, t_i where $i = (k, n, \alpha)$, to measure this elementary flow. We then also have infinitely many such times to correspond with our flows.

Using the multi-index notation $i = (k, n, \alpha)$ and (2.3.7), we may define

$$M_i \equiv \left(\hat{g}^{(k)}\xi_i\hat{g}^{(k)-1}\right)_k, \quad \xi_i \equiv \frac{1}{(\lambda-\lambda_k)^n}E_{\alpha\alpha}, \tag{2.4.1}$$

where the $(\ \ )_k$ indicates that we should take only the part expanded[11] about λ_k. We have also used $E_{\alpha\alpha}$ to denote the matrix with all zero entries except a one in the (α, α) entry.

This ξ_i matrix is precisely the $B_-^{(k)}$ matrix we described above as corresponding to our elementary flow. The change in notation from B to ξ is intended only to prevent confusion between the elementary flows and the B matrices considered in the previous section, which we shall define via linear combinations of these elementary flows.

[10]This term flow in this context refers to ∂_t, so that the dynamical flow is $\partial_t L$. The chosen polynomials then determine the dynamical flows (and visa versa) by determining the value of the commutator. These flows also admit a description in terms of an associated group and Lie structure which is not necessary here, though a terse discussion of this notion of flows can again be found in [1].

[11]We need not worry about taking the singular part since the $\hat{g}^{(k)}$ are jets of order $n_k - 1$ as in the previous section, so every term about λ_k in this definition is singular.

We now use the elementary time t_i associated with M_i to write the Lax equation

$$\partial_{t_i} L = [M_i, L]. \tag{2.4.2}$$

By constructing linear combinations of these elementary flows, we can construct any system in the integrable hierarchy. Application of the chain rule shows that general times are also constructed via linear combinations of these elementary times. So, our next challenge is to describe the structure of this hierarchy, which brings us to the following Proposition.

Proposition 3. The matrices M_i, M_j defining the time evolutions t_i, t_j satisfy the zero curvature condition

$$\partial_{t_i} M_j - \partial_{t_j} M_i - [M_i, M_j] = 0. \tag{2.4.3}$$

Remark. Before we prove this result, we shall first remark on two things. Firstly, this is called the zero curvature condition simply because it has the form of the mathematical generalization of the zero curvature condition satisfied by the electromagnetic tensor, $\mathrm{d}F = 0$. Secondly, this statement does not necessarily imply that $[M_i, M_j] = 0$ since the partial derivatives may not be zero. If we return to the definition (2.4.1), we will note that $\hat{g}^{(k)}$ may well be a function of all of the elementary times, and hence so may M_i. This condition does not, therefore, imply that the M_i and M_j commute, though it does actually imply that the dynamical flows $\partial_{t_i} L$ and $\partial_{t_j} L$ do commute. For the purposes of our treatment, however, the zero curvature condition is more greater consequence than the commutivity of elementary flows.

Proof. $\mathcal{L}$et $i = (k, n, \alpha)$ and $j = (k', n', \alpha')$. Diagonalizing L about λ_k by $L = \hat{g}^{(k)} A^{(k)} \hat{g}^{(k)-1}$ transforms (2.4.2) into

$$\partial_{t_j} \hat{g}^{(k)} = M_j \hat{g}^{(k)} + \hat{g}^{(k)} d_j^{(k)} \tag{2.4.4}$$

where $d_j^{(k)}$ is some unknown diagonal matrix. Now, by taking a ∂_{t_j} derivative of (2.4.1) and applying the relationship (2.4.4), we find

$$\partial_{t_j} M_i = \left[M_j, \hat{g}^{(k)} \xi^{(k)} \hat{g}^{(k)-1}\right]_k + \left(g^{(k)} \left[d_j^{(k)}, \xi_i\right] g^{(k)-1}\right)_k. \tag{2.4.5}$$

But since both $d_j^{(k)}$ and ξ_i are diagonal, the commutator in the second term must vanish. In precisely the same fashion, we obtain the same result but with the indices i and j switched,

$$\begin{aligned} \partial_{t_j} M_i &= \left[M_j, \hat{g}^{(k)} \xi_i \hat{g}^{(k)-1}\right]_k, \\ \partial_{t_i} M_j &= \left[M_i, \hat{g}^{(k')} \xi_j \hat{g}^{(k')-1}\right]_{k'}. \end{aligned} \tag{2.4.6}$$

If it is the case that $\lambda_k \neq \lambda_{k'}$, then we find the zero curvature condition immediately since M_j is regular at λ_k and only $M_i \equiv \left(\hat{g}^{(k)}\xi_i\hat{g}^{(k)-1}\right)_k$ contributes to the polar part of the commutator. Since we may make a similar argument about $\lambda_{k'}$,

$$[M_i, M_j] = [M_i, M_j]_k + [M_i, M_j]_{k'}. \tag{2.4.7}$$

If instead we suppose that $\lambda_k = \lambda_{k'}$, then we still have (2.4.6), but now the both of the singular parts are taken at λ_k. Since $M_i - \hat{g}^{(k)}\xi_i\hat{g}^{(k)-1} = \mathcal{O}(1)$ and $M_j - \hat{g}^{(k)}\xi_j\hat{g}^{(k)-1} = \mathcal{O}(1)$, it follows that $[M_i - \hat{g}^{(k)}\xi_i\hat{g}^{(k)-1}, M_j - \hat{g}^{(k)}\xi_j\hat{g}^{(k)-1}]_k = 0$ since any value that commutator might have had would be at least $\mathcal{O}(1)$, and so would be projected out when we take the polar part about λ_k.

This commutation relation, however, implies

$$[M_i, M_j] - [\hat{g}^{(k)}\xi_i\hat{g}^{(k)-1}, M_j]_k - [M_i, \hat{g}^{(k)}\xi_j\hat{g}^{(k)-1}]_k + [\hat{g}^{(k)}\xi_i\hat{g}^{(k)-1}, \hat{g}^{(k)}\xi_j\hat{g}^{(k)-1}]_k = 0, \tag{2.4.8}$$

but the final term vanishes since both ξ_i and ξ_j are diagonal, and so the zero curvature condition follows by the same argument as when the pole locations were distinct. □

With this Proposition, we have shown that elements of the integrability hierarchy pairwise satisfy the zero curvature constraint. This constraint then allows us to define the so called wave function, which will be the final stop in this development of concepts as we will then be in a position to define integrable field theories and dressing transformations, calling on what we have developed here for conceptual aids.

Definition 2. The wave function $\Psi(\lambda; t_1, t_2, \ldots)$, is a matrix function depending on all elementary times simultaneously and which satisfies

$$\partial_{t_i}\Psi = M_i\Psi, \quad \Psi(\lambda, t)\big|_{t=0} = 1. \tag{2.4.9}$$

We do not actually strictly require the normalization condition. If we do not impose the normalization, the wave function is merely defined up to right multiplication by a diagonal matrix. With this normalization, however, we fix the diagonal matrix and so the wave function would actually be unique.

Further, we shall remark that (2.4.9) is actually equivalent to the zero curvature condition, though we will not prove this, we will only point out that this equality ensures the integrability of the zero curvature condition in much the same way that requiring that $\nabla \times \mathbf{Q} = 0$ ensures that $\nabla F = \mathbf{Q}F$ has a solution in three dimensions for the normal, non-matrix, function F and vector field $\mathbf{Q}$.

§2.5 Integrable Field Theories

The theory developed above always assumed the systems of interest contained only a finite number of degrees of freedom in them. When we move to integrable field theories, we are forced to abandon this and adopt an infinitum of degrees of freedom, those contained in the values of our fields.

In this setting, if we are to consider a $1+1$ dimensional field theory, then the Lax equation (2.3.1) generalizes accordingly to the zero curvature condition

$$\partial_t U - \partial_x V - [V, U] = 0. \tag{2.5.1}$$

The precise justification for this generalization comes from the study of the coadjoint orbit and Lie structure associated with integrable systems. This associated group structure requires little alteration when transitioning to field theories, and so makes for a natural tool in generalizing the Lax equation. Though interesting in its own right, the theory of coadjoint orbits is not of use to us beyond providing (2.5.1) and so has been omitted. A detailed account of this theory may be found in [1].

For our purposes, we may either take (2.5.1) on faith, or we may observe the similarity between this condition and (2.4.3), where now the parameters in our field theory appear to be acting like the elementary times defined in the previous section. With this observation it should not be difficult to reconcile the sufficiency of (2.5.1) in defining the field theory in light of our analysis till now. Consider the following: we have taken great pains to point out that the pole structure of the integrable theory completely defines the model by informing us as to the number of allowed dynamical variables along with the conserved quantities when we diagonalize about each pole. It was the M matrix which defined the dynamical flows, and therefore the particular solution at which we are looking. Since, as remarked in §2.3, the pole structure of the M matrix is identical to the pole structure L possesses, the M matrix not only contains the information about the dynamical flows, but also implicitly contains the information required to specify the L matrix via the pole structure.

Therefore, when we move to integrable field theories, it is sufficient to specify the analogue of the M matrix since it effectively contains all information about our solution to the model. In our example of a $1+1$ field theory, however, we do not have a single flow which must be specified. Not only must we specify the flow in time, but we must also specify the flow in our spatial coordinate x. Therefore, we must actually have two analogues of the M matrix, here denoted by U and V. In the context of integrable field theories, the pair U, V is referred to as the Lax connection.

Just as with the elementary flows, this zero curvature condition allows us to define the wave function by

$$(\partial_x - U)\Psi(\lambda; x, t) = 0, \quad (\partial_t - V)\Psi(\lambda; x, t) = 0, \tag{2.5.2}$$

which converts the zero curvature condition to a pair of linear equations. As before, the wave function is defined up to right multiplication by a constant matrix, though we may fix this constant by imposing the normalization $\Psi(\lambda; 0, 0) = 1$. This wave function may be written symbolically be integrating the associated linear system as

$$\Psi(x,t) = \overleftarrow{\exp}\left[\int_\gamma (U\mathrm{d}x + V\mathrm{d}t)\right], \tag{2.5.3}$$

where γ is some path from the origin to the point (x,t), the zero curvature condition implying that this definition is actually independent of the chosen path γ. However, this is a path ordered exponential and, excepting very simple cases, is no more than a formal representation of the wave function which we cannot actually compute in practice. The notion that it *can* be written in such a form will become useful later on.

At this point we will end our general discussion on integrable field theories since we will need no more than this conceptual understanding. The interested reader is, however, referred to [1] for further discussion in this context or [5] for a more generic treatment. We end this section by presenting two well known integrable models to demonstrate concrete realizations of the notions.

The first example, and the simplest, is the non-linear sigma model in which U and V have only one simple pole at two different points, that is

$$U = \frac{1}{\lambda - 1} J_x, \quad V = -\frac{1}{\lambda + 1} J_t, \tag{2.5.4}$$

where J_x and J_t are valued in some Lie algebra[12]. By decomposing the zero curvature condition about the two poles and requiring that the condition holds for all λ, we find the two equations

$$\begin{aligned} \partial_t J_x - \frac{1}{2}[J_x, J_t] &= 0, \\ \partial_x J_t + \frac{1}{2}[J_x, J_t] &= 0. \end{aligned} \tag{2.5.5}$$

If we now take the difference of these two equations, we find $[\partial_t + J_t, \partial_x + J_x] = 0$. This condition implies that J_x and J_t are pure gauge, that is, equal to zero up to a gauge transformation. This implies that there exists a function g such that $J_x = g^{-1}\partial_x g$ and $J_t = g^{-1}\partial_t g$, though for our purposes, we need only substitute this ansatz into $[\partial_t + J_t, \partial_x + J_x] = 0$ to show that it is a solution. Since this is a single partial differential equation in a two variables,

[12] One possible choice is for these to be valued in a 2×2 representation of Lie algebra associated with the group SU(2). In this case, J_x and J_t would be formed by linear combinations of the Pauli matrices.

we would expect our solution to be determined up to one function's worth of freedom, this is no more than the generalization of having a constant's worth of freedom in an ordinary differential equation which sits in one dimension. Therefore, this pure gauge solution is general.

If instead we sum the equalities in (2.5.5), we find that $\partial_t J_x + \partial_x J_t = 0$. It then follows that

$$\partial_t(g^{-1}\partial_x g) + \partial_x(g^{-1}\partial_t g) = 0. \tag{2.5.6}$$

In the specific case of the non-linear sigma model, we have therefore reduced the zero curvature condition to a single non-linear partial differential equation in the gauge field g. Specifying the matrix g is then equivalent to specifying an entire solution to the model.

The second example we will present here will be the Sinh-Gordon model. This model again has only two poles, but this time one of the two poles sits at $\lambda = 0$ and the other at $\lambda = \infty$. For this model, we require that, in the light-cone coordinates $x + \pm = x \pm t$, $U(\lambda; x_\pm)$ has a simple pole at $\lambda = 0$ and $V(\lambda; x_\pm)$ has a simple pole at $\lambda = \infty$. We may then consider the 2×2 system

$$\begin{aligned} (\partial_{x_+} - U)\Psi = 0, \quad U = U_0 + \lambda^{-1}U_1, \\ (\partial_{x_-} - V)\Psi = 0, \quad V = V_0 + \lambda V_1. \end{aligned} \tag{2.5.7}$$

The matrices U_i and V_i are taken to be traceless and so we have 12 free parameters in total. We may reduce this number by additionally imposing a symmetry condition based on a discrete group, say $\mathbb{Z}_2$ for the sake of example. Then we demand that Ψ be invariant under the action

$$\Psi(\lambda) \to \sigma_z \Psi(-\lambda)\sigma_z, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.5.8}$$

so σ_z is a $\mathbb{Z}_2$ representation. It then follows by applying this condition to (2.5.7) that U and V must also satisfy this symmetry, and we are still able to perform a gauge transformation by some gauge g independent of λ so that it preserved the pole structure, and commuting with the $\mathbb{Z}_2$ action to leave the symmetry satisfied. So, g must be diagonal.

Utilizing all of the constraints, we find that U and V must have the form

$$U = \begin{pmatrix} u_0 & \lambda^{-1}u_1 \\ \lambda^{-1}u_2 & -u_0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \lambda v_1 \\ \lambda v_2 & 0 \end{pmatrix}. \tag{2.5.9}$$

In this gauge, the zero curvature equation reduces to

$$\begin{aligned} \partial_{x_-} u_0 - u_1 v_2 + v_1 u_2 = 0, \\ \partial_{x_-} u_1 = 0, \quad \partial_{x_-} u_2 = 0, \\ \partial_{x_+} v_1 - 2v_1 u_0 = 0, \quad \partial_{x_+} v_2 + 2v_2 u_0 = 0. \end{aligned} \tag{2.5.10}$$

The second line of equations here implies that $u_1 = \alpha(x_+),\ u_2 = \beta(x_+)$ for some arbitrary functions α and β. If we set $u_0 = \partial_{x_+}\varphi$, then from the third line we find $v_1 = \gamma(x_-)e^{2\varphi}$ and $v_2 = \delta(x_-)e^{-2\varphi}$ for some arbitrary functions γ and δ. With this, the first line then becomes

$$\partial_{x_+}\partial_{x_-}\varphi + \beta(x_+)\gamma(x_-)e^{2\varphi} - \alpha(x_+)\delta(x_-)e^{-2\varphi} = 0, \tag{2.5.11}$$

but the arbitrary functions $\alpha, \beta, \gamma,$ and δ are irrelevant here since they can simply be absorbed into a redefinition of the field φ and a change in the coordinates $x_\pm$. Doing this, we find

$$\partial_{x_+}\partial_{x_-}\varphi + \sinh(2\varphi) = 0, \tag{2.5.12}$$

hence the sinh in the name Sinh-Gordon equation.

Solutions to a similar system, the Cosh-Gordon equation, become important in certain applications in string theory. Solutions to the Cosh-Gordon equation were found in terms of Riemann Theta functions by Ishizeki, Kruczenski, and Ziama in [9] and [11]. In certain limits, the system of interest discussed in §4 reduces to the Cosh-Gordon equation, making this an interesting point of comparison.

§3 Dressing Transformations

Since the spatial and temporal variables of a $1+1$ integrable field theory are mathematically indistinguishable from the elementary times, we would expect there to also exist a hierarchy of solutions U and V to the zero curvature condition of our integrable field theory. If there was a method by which we could map from one solution, U, V to some other pair in the hierarchy, say U^g, V^g, then the new pair must also be a solution to our model since the hierarchy of elementary times was constructed specifically to preserve the pole structure of the model. Thus, such a mapping would allow us to obtain new solutions from old one, though we might be careful to check that we have not obtained a trivially "new" solution which is related to our original solution via some kind of coordinate change. Such a solution would not really be new and so any effort expended finding a solution of this type would be in vain.

Dressing transformations represent such a mapping within the integrable hierarchy, and more importantly, these transformations are capable of resulting in solutions which are genuinely new and not simply disguised by some kind of coordinate or gauge transformation.

To better sketch this notion of nontrivial transformations between solutions, we shall first discuss the notion of Bäcklund transformations. The dressing transformations we will be discussing in the ensuing material are a special kind of Bäcklund transformation.

Bäcklund transformations can generally describe a transformation from a solution of one differential equation to a solution of some other differential equation though the canonical

example maps to and from the same differential equation and so is called an auto-Bäcklund transformation. This example is actually a familiar one: the Cauchy-Riemann equations of complex analysis,

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v, \tag{3.1}$$

which relate the real and imaginary components of a holomorphic function $f(z) = u(x, y) + iv(x, y)$, $x, y \in \mathbb{R}$. It is a well known fact that if u and v satisfy the Cauchy-Riemann equations, then both u and v also satisfy the Laplace equation,

$$\partial_x \partial_x u + \partial_y \partial_y u = 0, \tag{3.2}$$

which is easily verified by taking derivatives of (3.1) and using the commutivity of partial derivatives. It then follows that if we have a particular solution of the Laplace equation, say u, then the auto-Bäcklund transformation (3.1) will generate a new solution v by solving a pair of linear differential equations.

The Sinh-Gordon equation described in the previous section also happens to have an auto-Bäcklund transformation. Though there is no general method to constructing Bäcklund transformations, we can say that if φ satisfies a second order, potentially non-linear, partial differential equation, then the transform to a new solution $\hat{\varphi}$ must have the form

$$\partial_x \hat{\varphi} = P(\hat{\varphi}, \varphi, \partial_x \varphi, \partial_y \varphi), \quad \partial_y \hat{\varphi} = Q(\hat{\varphi}, \varphi, \partial_x \varphi, \partial_y \varphi), \tag{3.3}$$

for some functions P and Q. These functions are constrained by the condition $\partial_x \partial_y \hat{\varphi} = \partial_y \partial_x \hat{\varphi}$, and in the case of the Sinh-Gordon equation read as

$$\partial_x(\varphi + \hat{\varphi}) = 2m\lambda^{-1}\sinh(\varphi - \hat{\varphi}), \quad \partial_y(\varphi - \hat{\varphi}) = 2m\lambda\sinh(\varphi + \hat{\varphi}), \tag{3.4}$$

where λ is an arbitrary parameter. So, if we already have a solution to the Sinh-Gordon equation, then we are able to obtain a new solution to the equation by solving two first order differential equations rather than the full Sinh-Gordon equation. Now, in this example the Bäcklund transformation hardly buys us anything at all since we are still stuck with a pair of non-linear differential equations[13], but the idea that we can generate non-trivially new solutions from old ones is what we shall pursue in this chapter.

The final thing we must discuss before beginning our development of dressing transformations is the Riemann-Hilbert problem. For our purposes here, we need only describe the Riemann-Hilbert it as a factorization problem. Given an invertible matrix-valued function

[13]It is possible to convert this to a linear system in terms of the U and V in the previous section, but we will not develop this here. The development is presented in [1] and amounts to setting $e^{\hat{\varphi}} = \frac{u}{v} e^{\varphi}$, expanding (3.4) in terms of u and v, and identifying the matrices U and V.

$g(\lambda)$ defined on some closed contour Γ in the complex plane, can we find a factorization $g = g_-^{-1} g_+$ such that g_- is analytic outside the contour and g_+ is analytic inside the contour. The existence of such a factorization is ensured by the following Theorem

Theorem 4. Let Γ be the closed contour $|\lambda| = 1$ in the λ-plane and $g(\lambda)$ be a matrix defined on Γ. There exist two matrices $g_\pm(\lambda)$ such that $g_+(\lambda)$ is analytic within Γ and $g_-(\lambda)$ is analytic outside Γ and $\det g_\pm \neq 0$ within their respective domains of definition. These matrices additionally satisfy

$$g(\lambda) = g_-^{-1}(\lambda)\Lambda(\lambda)g_+(\lambda), \tag{3.5}$$

where $\Lambda(\lambda)$ is a diagonal matrix with entries of the form λ^{k_i}. The k_i are integers uniquely determined up to order by $g(\lambda)$ and referred to as the indices. This solution to the factorization problem is unique if we require $g_-(\lambda)|_{\lambda=\infty} = 1$.

Remark. Since the k_i are required to be integers, then if $g(\lambda)$ is "close" to the identity, a statement which we might make precise in terms of the Lie algebra generating g, then we may take the indices to be zero and so in such a case we may neglect the matrix Λ all together, a fact we will be promptly using to our advantage.

As a second remark, we will also point out that the restriction to only unit radius circular contours is really no restriction at all since we can just transform our complex parameter to obtain some other contour. Since this Theorem requires there to be no poles in any of these matrices within their respective domains of definition, the fundamental group of the λ-plane is trivial and so our choice in contour truly is unrestricted.

We will point out that this Theorem only asserts that the solution to the factorization problem exists, it does not give any methods by which we might find the factorization and in general, finding the solution is simply not possible since it amounts to a very large non-linear algebraic problem. We will find, however, that there are certain specialized cases in which we can find the factorization, and that these are all we require.

§3.1 The Dressing Group

Keeping in mind that our ultimate goal is to use the solution to the Riemann-Hilbert problem to generate an auto-Bäcklund transformation for the zero curvature condition. To this end, we begin by defining the matrix function

$$\theta(\lambda; x, t) = \Psi(\lambda; x, t)g(\lambda)\Psi^{-1}(\lambda; x, t), \tag{3.1.1}$$

where Ψ is the wave function associated with the zero curvature condition of the field theory and $g(\lambda)$ is a regular, invertible matrix defined on some contour Γ. It does not particularly

matter that contour is chosen, the only restriction is that the poles of our Lax connection should not lie on contour. We could quite easily take Γ to be the union of small contours surrounding each pole which may aid in visualizing the structure of the problem, though this is by no means necessary. Strictly speaking, the matrix $g(\lambda)$ must be a member of the loop group associated with the pole structure of the field theory. This, however, is not going to be particularly important to us, and would again require us to have treated the relationship between integrable theories and their associated group structure. What is important to us, is that the loop group has a Lie structure, and we are therefore able to reasonably require $g(\lambda)$ to be sufficiently close to the identity so that $\theta(\lambda; x, t)$ is also near the identity and we may neglect the indices in factoring θ as

$$\theta(\lambda; x, t) = \theta_-^{-1}(\lambda; x, t)\theta_+(\lambda; x, t), \tag{3.1.2}$$

where the $+$ and $-$ functions are defined inside and outside of Γ, respectively. Additionally, the matrix $g(\lambda)$ itself admits a factorization $g(\lambda) = g_-^{-1}(\lambda)g_+(\lambda)$.

With this we are in a position to state the fundamental Theorem of dressing transformations which presents both the form of the transformation and asserts that the dressing does indeed map us between valid solutions.

Theorem 5. If Ψ is the normalized wave function associated with the Lax connection U, V, then the function

$$\Psi^g(\lambda; x, t) = \theta_\pm(\lambda; x, t)\Psi(\lambda; x, t)g_\pm^{-1}(\lambda) \tag{3.1.3}$$

is defined for all λ on the contour Γ on which $g(\lambda)$ is defined. Within Γ, except at the essential singularities λ_k, this function extends to a function Ψ_+^g. Outside of Γ, the function extends to a function Ψ_-^g. On the contour Γ, we have ${\Psi_-^g}^{-1}\Psi_+^g|_\Gamma = 1$, so the $\Psi_\pm^g$ define a unique function Ψ^g normalized by $\Psi^g(\lambda, 0, 0) = 1$ which is defined everywhere but the essential singularities λ_k. Additionally, Ψ^g is the solution of the linear system associated with the zero curvature condition whose Lax connection is given by

$$\begin{aligned} U^g(\lambda; x, t) &= \theta_\pm U\theta_\pm^{-1} + (\partial_x\theta_\pm)\theta_\pm^{-1}, \\ V^g(\lambda; x, t) &= \theta_\pm V\theta_\pm^{-1} + (\partial_t\theta_\pm)\theta_\pm^{-1}. \end{aligned} \tag{3.1.4}$$

Further, U^g and V^g are meromorphic functions on the whole complex λ plane with the same analytic structure as the original Lax connection U, V.

Proof. We will begin by observing that the definitions of $g_\pm$ and $\theta_\pm$ imply that

$$\theta_+(\lambda; x, t)\Psi(\lambda; x, t)g_+^{-1}(\lambda) = \theta_-(\lambda; x, t)\Psi(\lambda; x, t)g_-^{-1}(\lambda) \tag{3.1.5}$$

holds for all $\lambda \in \Gamma$, so (3.1.3) defines a unique function Ψ^g on Γ. The extension to the inside and outside of Γ is therefore also unique since we do not have any conflicting definitions in either region since we define the extensions $\Psi^g_\pm$ by

$$\Psi^g_\pm = \theta_\pm \Psi g^{-1}_\pm. \tag{3.1.6}$$

These functions have the same essential singularities as Ψ at the points λ_k since g is regular by definition and $\theta_\pm$ must also be regular since, when we defined θ, the definition included a product of Ψ and Ψ^{-1}. Since whenever Ψ has a pole of a certain order, Ψ^{-1} must also have a zero of the same order and visa versa, it must be the case that θ is regular, and therefore so are the $\theta_\pm$. Since we have already argued that the functions (3.1.6) are equal on the contour Γ, it follows immediately that $\Psi^{g\ -1}\Psi^g_+|_\Gamma = 1$.

If we now take (3.1.6) in conjunction with the linear system associated with the zero curvature condition,

$$(\partial_x - U^g_\pm)\Psi^g_\pm(\lambda; x, t) = 0, \quad (\partial_t - V^g_\pm)\Psi^g_\pm(\lambda; x, t) = 0, \tag{3.1.7}$$

we may rearrange terms, and utilize the fact that Ψ satisfied its own Lax connection with components U and V to find

$$\begin{aligned} U^g_\pm &= \partial_x \Psi^g_\pm \Psi^{g\ -1}_\pm = \partial_x \theta_\pm \theta^{-1}_\pm + \theta_\pm U \theta^{-1}_\pm, \\ V^g_\pm &= \partial_t \Psi^g_\pm \Psi^{g\ -1}_\pm = \partial_t \theta_\pm \theta^{-1}_\pm + \theta_\pm V \theta^{-1}_\pm. \end{aligned} \tag{3.1.8}$$

Since both of the $\Psi^g_\pm$ coincide on Γ, it follows that $U^g_\pm$ and $V^g_\pm$ do as well, and hence the pair $U^g_\pm, V^g_\pm$ defines a Lax connection U^g, V^g on the entire λ-plane. Since, as we have already argued, the $\theta_\pm$ are regular in their respective domains of definition, the U^g and V^g inherit the pole structure of U and V. □

There are several things that must be remarked on here. First, we will point out that we only showed that the new Lax connection, U^g, V^g, satisfies the associated linear system, and therefore the zero curvature condition, and that the new connection has the same pole structure as the original. A conceptual understanding of why this is all we need to call U^g, V^g a new solution to our field theory is, in large part, why we took such great pains to develop the theory of integrable systems in §2.

With the understanding we have developed to this point, we know that it is sufficient to specify the pole structure to specify the model. The U and V matrices merely specify the flows representing a particular solution within the model given by the pole structure. It follows then that any pair U^g, V^g which has same pole structure as specified a priori by the model and which satisfies the zero curvature condition *must* be precisely what we mean by another solution to the system.

Lacking this background, it is hard to see how (3.1.8) can define a new solution to our system since the same reasoning would not apply to the differential systems to which we are most accustomed. For example, when solving the Schrödinger equation for some given potential, we have a general prescription for writing down the solution, though our ability to actually do so is, of course, severely limited by the nature of the potential involved. For example, if we are working in the position basis, we might separate variables, then solve the time independent equation to find the allowable energy values, so on and so forth. What we have done in proving only that U^g, V^g has the same pole structure might be seen as more akin to writing down some function and then declaring it to "solve" the Schrödinger equation for *some* potential, maybe just not the one we were worried about to start with. A statement like this can, and probably should, be called nonsense, but we are careful to notice *why* it is nonsense.

When we seek solutions to the Schrödinger equation, we really mean to say that we seek solutions to the equation for a given potential, because the chosen potential is what specifies the model we are using for the system of interest, along with, perhaps, the number of particles to determine the number of kinetic terms present. So by saying that we can write down a function which satisfies the Schrödinger equation for some other potential, we are really saying that we can write down a solution for some model other than the one we are trying to solve.

What we have done here is fundamentally different since it is not a function that defines the model we are working in, but rather the pole structure. So, this dressing transformation we have defined does not simply produce an arbitrary solution that works for *some* system, but rather a solution which is valid specifically for the system we are modeling. In the Schrödinger equation analogue, this would be more akin to producing some transformation which, given an energy eigenfunction, transforms us to a new energy eigenfunction for the same system.

We should also point out that this is similar in nature, though not precisely the same as, the notion of auto-Bäcklund transformations discussed earlier. We will note, however, that the dressing transformations, unlike the Bäcklund transformations, require that we solve the Riemann-Hilbert problem (3.1.2) rather than a pair of first order differential equations. This could be argued to be a good or a bad thing. On the one hand, the Riemann-Hilbert problem is a purely algebraic one rather than a differential one as the Bäcklund transformations are. On the other hand, however, the Riemann-Hilbert problem is a highly non-linear one and may well not be solvable if we are to approach the problem as a polynomial system in the matrix entries, further complicated by the requirements of the λ dependence. The up side to this dressing approach, as we will see in the following section, is that there is a way around the Riemann-Hilbert problem so we need not actually solve it in a non-trivial case.

Now, before we end this section, there is one final thing to point out. This collection

of transformations is called the dressing group for a reason. The collection of dressing transformations $\Psi \to \Psi^g$ do indeed form a group under composition. The group operation is not particularly difficult to define, nor is it particularly important. The fact that these transformations are closed under composition, however, is important to know in order to assert the existence of so-called multi-soliton solutions, which we shall touch on at a later time.

§3.2 Soliton Solutions

In discussing the dressing transformations, and more generally, in the Riemann-Hilbert problem, the assumption $\det \theta_\pm \neq 0$ we carried along. This condition, however, does little more than ensure the uniqueness of the factorization, and therefore the transformation. If we take care, it is possible to relax this condition and allow $\theta_\pm$ to have zero determinants at isolated points. This generalized factorization problem is referred to as the Riemann-Hilbert problem with zeroes and is precisely the generalization that brings us from dressing transformations to the theory of soliton solutions.

What follows immediately here is related first and foremost to the Riemann-Hilbert problem with zeroes, and so we have changed notation to avoid confusion with either the preceding or ensuing material.

Definition 3. We will say that a matrix function $\eta(\lambda)$ has a zero at a point λ_0 is $\det \eta(\lambda_0) = 0$ and in the vicinity of λ_0,

$$\eta(\lambda) = F_0 + (\lambda - \lambda_0)F_1 + \mathcal{O}(\lambda - \lambda_0)^2, \quad \eta^{-1}(\lambda) = \frac{1}{\lambda - \lambda_0}C_0 + C_1 + \mathcal{O}(\lambda - \lambda_0), \qquad (3.2.1)$$

that is, η is regular at λ_0 but η^{-1} is not.

In particular, if a $\eta(\lambda)$ has a zero, then since $\eta(\lambda)\eta^{-1}(\lambda) = \eta^{-1}(\lambda)\eta(\lambda) = 1$, we find that $F_0C_0 = C_0F_0 = 0$, and $C_0F_1 + C_1F_0 = 1$. In particular, this tells us that we must have

$$\text{Ker } F_0 = \text{Im } C_0, \quad \text{Ker } C_0 = \text{Im } F_0. \qquad (3.2.2)$$

Much as with the dressing transformations, we will let Γ be some closed contour in the λ-plane. Though we have not yet made contact again with the Lax system, once we do we shall again require that the poles of the Lax connection do not lie on Γ, but the contour as otherwise unconstrained.

If $h(\lambda)$ is some matrix function defined on Γ, the Riemann-Hilbert problem with zeroes is to find a factorization

$$h(\lambda) = \eta_-^{-1}(\lambda)\eta_+(\lambda) \tag{3.2.3}$$

where $\eta_+(\lambda)$ is analytic within Γ and has some number N of zeroes located at the points $\mu_1, \ldots, \mu_N$, and $\theta_-^{-1}(\lambda)$ is analytic outside Γ with N zeroes at the points $\lambda_1, \ldots, \lambda_N$. We will emphasize that it is the matrix θ_-^{-1} which is analytic and has the zeroes outside the contour, not θ_-.

With the Riemann-Hilbert problem with zeroes defined, we are in a position to prove the following Theorem which informs us in precisely what way the uniqueness of the factorization fails with the relaxation of the non-zero determinant condition.

Theorem 6. Define the two collections of subspaces,

$$V_n = \text{Im } \theta_-^{-1}(\lambda)|_{\lambda=\lambda_n}, \quad W_n = \text{Ker } \theta_+(\lambda)|_{\lambda=\mu_n}, \tag{3.2.4}$$

for all $n \leq N$. If we specify the subspaces V_n and W_n, then we have also uniquely specified the soliton to the Riemann-Hilbert factorization problem with zeroes up to left multiplication by a constant matrix.

Proof. We prove this Theorem by supposing that there are two solutions to the Riemann-Hilbert problem with zeroes, (3.2.3), $\eta_\pm$ and $\tilde{\eta}_\pm$. With these, we may define the function $\chi = \tilde{\eta}_-\eta_-^{-1} = \tilde{\eta}_+\eta_+^{-1}$, which is meromorphic and well-defined on the entire λ-plane since the products must agree on Γ by definition and when we are off the contour, we may simply choose the product which is actually defined. This function potentially has poles located at the points λ_n and μ_n.

If we expand near a pole, say μ_n, then we have

$$\begin{aligned} \eta_+(\lambda) &= F_n + \mathcal{O}(\lambda - \mu_n), \quad \eta_+^{-1}(\lambda) = \frac{1}{\lambda - \mu_n} C_n + \mathcal{O}(1), \\ \tilde{\eta}_+(\lambda) &= \tilde{F}_n + \mathcal{O}(\lambda - \mu_n), \quad \tilde{\eta}_+^{-1}(\lambda) = \frac{1}{\lambda - \mu_n} \tilde{C}_n + \mathcal{O}(1), \end{aligned} \tag{3.2.5}$$

since we are working with solutions with zeroes. Working in the vicinity of μ_n so that (3.2.5) is valid, we utilize (3.2.2) and (3.2.4) to write $W_n = \text{Ker } F_n = \text{Im } C_n$, and $W_n = \text{Ker } \tilde{F}_n = \text{Im } \tilde{C}_n$. It therefore follows that $\chi = \frac{1}{\lambda - \mu_n} \tilde{F}_n C_n + \mathcal{O}(1)$ is regular because clearly $\tilde{F}_n C_n = 0$.

The same analysis holds at λ_n, so $\chi(\lambda)$ is regular everywhere, and hence is a constant since χ is meromorphic and also does not have a pole at infinity. Thus, $\eta_\pm = \tilde{\eta}_\pm$ up to left multiplication by a constant matrix. □

This Theorem tells us that, in determining the factorization, it is sufficient to specify the images and kernels about each pole, (3.2.4). It is important to note, however, that we have proved only sufficiency. We have not proved that this is also a necessary condition, though it appears to be quite close to being one. This is a point we will return to at a later time.

Now, to this point, relaxing the determinant condition has only weakened our notion of uniqueness for the factorization. We still have not circumvented main problem in actually generating a dressing transform: how do we actually solve the Riemann-Hilbert problem when it presents such a large system of non-linear algebraic equations? This next Theorem provides just such an explicit solution for the problem with zeroes.

Theorem 7. Suppose that η_+ and η_-^{-1} comprise a solution to the Riemann-Hilbert problem (3.2.3) with zeroes at μ_N and λ_N, respectively. Further, fix $W_N = \text{Ker } \eta_+(\mu_N)$ and $V_N = \text{Im } \eta_-^{-1}(\lambda_N)$. Let $\tilde{\eta}_\pm$ be a solution to the same Riemann-Hilbert problem, with or without zeroes, which has no zeroes at μ_N and λ_N. If $\tilde{\eta}_\pm$ is a solution with zeroes, then $\eta_\pm$ must also share those zeroes. Then

$$\begin{aligned} \eta_+(\lambda) &= \chi_0^{-1}\left(1 - \frac{\mu_N - \lambda_N}{\lambda - \lambda_N} P_N\right)\tilde{\eta}_+(\lambda), \\ \eta_-^{-1}(\lambda) &= \tilde{\eta}_-^{-1}(\lambda)\left(1 - \frac{\lambda_N - \mu_N}{\lambda - \mu_N} P_N\right)\chi_0, \end{aligned} \tag{3.2.6}$$

where χ_0 is a constant matrix and P_N is a projection matrix such that

$$\text{Im } P_N = \tilde{\eta}_+(\mu_N)W_N, \quad \text{Ker } P_N = \tilde{\eta}_-(\lambda_N)V_N. \tag{3.2.7}$$

Proof. We begin by again introducing the matrix $\chi(\lambda) = \tilde{\eta}_-\eta_-^{-1} = \tilde{\eta}_+\eta_+^{-1}$ as we did in the proof of the previous Theorem. This is again a meromorphic function in the entire complex λ-plane with possible simple poles[14] at λ_N and μ_N.

Since these are the only two poles, one in χ and one in χ^{-1}, we are free to expand them globally about their poles by

$$\chi = \chi_0 + \frac{1}{\lambda - \mu_N}\chi_1, \quad \chi^{-1} = \chi_0' + \frac{1}{\lambda - \lambda_N}\chi_1'. \tag{3.2.8}$$

[14]If we were to relax the assumption that the poles did not have distinct locations, then it would be possible for us to pick up higher order pole terms. This generalization would complicate this proof and is not necessary for our purposes. However, this generalization can come up in practice such as in generating the multi-solitons to dress the giant magnon, [13]. In this instance, it was sufficient to make the two projection matrix we have called P_N orthogonal to $\tilde{\eta}$, which cancels the higher order pole terms which would otherwise arise.

By now imposing the condition $\chi(\lambda)\chi^{-1}(\lambda) = 1$, which must hold for all λ, we find

$$1 = \chi_0\chi_0' + \frac{\chi_0\chi_1'}{\lambda - \lambda_N} + \frac{\chi_1\chi_0'}{\lambda - \mu_N} + \frac{\chi_1\chi_1'}{(\lambda - \mu_n)(\lambda - \lambda_N)}, \tag{3.2.9}$$

which immediately implies that $\chi_0 = {\chi_0'}^{-1}$. By taking the residue of (3.2.9) at each of its poles, we also find

$$\begin{aligned} 0 &= \chi_1 \left(\chi_0' + \frac{\chi_1'}{\mu_N - \lambda_N} \right), \\ 0 &= \left(\chi_0 + \frac{\chi_1}{\lambda_N - \mu_N} \right) \chi_1'. \end{aligned} \tag{3.2.10}$$

By adding these two equations, we find $\chi_1\chi_0' = -\chi_0\chi_1'$ and use the first equation of (3.2.10) to write $\chi_1\chi_1' = (\lambda_N - \mu_N)\chi_1\chi_0'$. If we now define the suggestively named matrix

$$P_N = \frac{\chi_0\chi_1'}{\lambda_N - \mu_N} = -\frac{\chi_1\chi_0'}{\lambda_N - \mu_N}. \tag{3.2.11}$$

It is simple to show that $P_N^2 = P_N$ with the relationships developed to this point, so P_N is a projection matrix. Rearranging the parameterizations (3.2.8), we now find

$$\chi(\lambda) = \left(1 - \frac{\lambda_N - \mu_N}{\lambda - \mu_N} P_N \right) \chi_0, \quad \chi^{-1}(\lambda) = \chi_0^{-1} \left(1 - \frac{\mu_N - \lambda_N}{\lambda - \lambda_N} P_N \right). \tag{3.2.12}$$

From this form, the result (3.2.6) follows immediately.

All that remains to be shown at this point is that the image and kernel conditions are correct. By Theorem 6 above, the specification of the subspaces must also specify the factorization up to right multiplication by a constant matrix, and since P_N is the only ambiguous part of the form (3.2.6), this implies that the subspaces must determine P_N.

So, we simply write down the definition of the spaces, $W_N = \text{Ker } \eta_+(\mu_N) = \text{Ker } [(1 - P_N)\tilde{\eta}_+(\mu_N)]$ so that $(1 - P_N)\tilde{\eta}_+(\mu_N)W_N = 0$ or $\tilde{\eta}_+(\mu_N)W_N = \text{Ker } (1 - P_N) = \text{Im } P_N$, as desired.

In nearly the same fashion, we may write $V_N = \text{Im } \eta_-^{-1}(\lambda_N) = \text{Im } [\tilde{\eta}_-^{-1}(\lambda_N)(1 - P_N)]$. This of course implies that if $\mathbf{x}$ is a vector in the space acted on by our matrices, and $\mathbf{y} = \tilde{\eta}_-^{-1}(\lambda_N)(1 - P_N)\mathbf{x}$, then $\mathbf{y} \in V_N$. However, since $\tilde{\eta}_-(\lambda_N)$ is invertible at λ_N, we may just as well write this as $\tilde{\eta}_-(\lambda_N)V_N = \text{Im } (1 - P_N)$, which as before is equivalent to $\text{Ker } P_N = \tilde{\eta}_-(\lambda_N)V_N$. Thus, the Theorem is proven. □

Before stating this Theorem, we commented that this would provide us with a method for eluding the task of actually solving the Riemann-Hilbert problem. However, this theorem

requires a factorization in order to generate a new one. Though this factorization is allowed to have zeroes of its own, if we only build factorizations via this Theorem, then no matter how many zeroes we might have, we can always recurse backwards though applications of this Theorem to find that at some point we needed to supply a Riemann-Hilbert factorization without zeroes.

The key to applying this Theorem, however, is that it generate new factorizations from old ones in a non-trivial manner. If we were to solve some trivial factorization problem, say $1 = \tilde{\eta}_-^{-1}(\lambda)\tilde{\eta}_+(\lambda)$, which clearly has the unique solution $\tilde{\eta}_\pm = 1$, we can use this factorization with Theorem 7 to generate factorizations with zeroes which are most certainly non-trivial ones. This is what we meant when we asserted that this Theorem would allow us to circumvent the monumental task of solving the Riemann-Hilbert problem.

With our newfound ability to *actually* write down non-trivial solutions to the Riemann-Hilbert problem with zeroes, all we require is an extension of the methods of dressing transformations to factorizations with zeroes.

Theorem 8. Suppose that U and V are the components of a Lax connection, referred to as the vacuum solution, with n poles and the associated wave function, $\Psi(\lambda; x, t)$. We also suppose that $g(\lambda) = g_-^{-1}(\lambda)g_+(\lambda)$ is a loop group matrix defined on a contour Γ, which does not pass through any of the poles of the given Lax connection, and if factored according to the Riemann-Hilbert problem *without* zeroes. If we now fix the vector spaces $V_k(0)$ and $W_k(0)$, $k \leq n$, then we may define the spaces

$$V_k(x,t) = \Psi(\lambda_k; x, t)V_k(0), \quad W_k(x,t) = \Psi(\mu_N; x, t)W_k(0), \tag{3.2.13}$$

to define the Riemann-Hilbert factorization with zeroes at $\lambda_1, \ldots, \lambda_N$ and $\mu_1, \ldots, \mu_N$ of the matrix $\Psi^{-1}g\Psi = \theta_-^{-1}\theta_+$.

The transformation $\Psi \to \Psi^g$, where

$$\Psi^g = \theta_\pm \Psi g_\pm^{-1}, \tag{3.2.14}$$

is a dressing transformation. That is, it defines a new Lax connection with the same analytic structure as the connection U, V.

Proof. We begin with the linear system

$$(\partial_x - U(\lambda; x, t))\Psi = 0, \quad (\partial_t - V(\lambda; x, t))\Psi = 0, \tag{3.2.15}$$

and dress it with the transformation $\Psi \to \Psi^g$ defined by (3.2.14) and find

$$\begin{aligned} U^g(\lambda; x, t) &= \theta_\pm U\theta_\pm^{-1} + \partial_x\theta_\pm\theta_\pm^{-1}, \\ V^g(\lambda; x, t) &= \theta_\pm V\theta_\pm^{-1} + \partial_t\theta_\pm\theta_\pm^{-1}, \end{aligned} \tag{3.2.16}$$

as in (3.1.4). In general, this dressing allows for the possibility of poles at the points μ_n, λ_n, but this would clearly be a change in the analytic structure of the connection. Therefore, we must require that the residues of these poles vanish so that no new pole will be added to the analytic structure. At $\lambda = \mu_n$, we have

$$\theta_+(\lambda) = A_n + \mathcal{O}(\lambda - \mu_n), \quad \theta_+^{-1}(\lambda) = \frac{1}{\lambda - \mu_n} B_n + \mathcal{O}(1), \tag{3.2.17}$$

which we may use in conjunction with (3.2.16) and the identity $\partial\theta\theta^{-1} = -\theta\partial\theta^{-1}$ to write

$$\begin{aligned} U^g &= \frac{1}{\lambda - \mu_n} A_n U B_n - \frac{1}{\lambda - \mu_n} A_n \partial_x B_n + \mathcal{O}(1), \\ V^g &= \frac{1}{\lambda - \mu_n} A_n V B_n - \frac{1}{\lambda - \mu_n} A_n \partial_t B_n + \mathcal{O}(1). \end{aligned} \tag{3.2.18}$$

Now, the requirement that Res $U^g|_{\lambda=\mu_n} = 0$ and Res $V^g|_{\lambda=\mu_n} = 0$ may be written in the form

$$A_n(\partial_x - U|_{\lambda=\mu_n})B_n = 0, \quad A_n(\partial_t - V|_{\lambda=\mu_n})B_n = 0. \tag{3.2.19}$$

From this condition, we utilize the definition (3.2.4) and statement (3.2.2) to conclude that the space $W_n = \text{Ker } A_n = \text{Im } B_n$ must be invariant under the action of the operators $\partial_x - U|_{\lambda=\mu_n}$ and $\partial_t - V|_{\lambda=\mu_n}$. To elucidate this point, we first observe that the first equation of (3.2.19) implies Im $[(\partial_x - U|_{\lambda=\mu_n})B_n] = \text{Ker } A_n$. But since (3.2.2) implies Ker $A_n = \text{Im } B_n$, it must be the case that $\partial_x - U|_{\lambda=\mu_n}$ is an automorphism of Im B_n, and thus that W_n is invariant under $\partial_x - U|_{\lambda=\mu_n}$. A nearly identical argument shows that W_n is invariant under $\partial_t - V|_{\lambda=\mu_n}$.

In much the same way, at the point $\lambda = \lambda_n$, we may write

$$\theta_-^{-1}(\lambda) = C_n + \mathcal{O}(\lambda - \lambda_n), \quad \theta_-(\lambda) = \frac{1}{\lambda - \lambda_n} D_n + \mathcal{O}(1), \tag{3.2.20}$$

which allows us to set the residues to zero and write

$$D_n(\partial_x - U|_{\lambda=\lambda_n})C_n = 0, \quad D_n(\partial_t - V|_{\lambda=\lambda_n})C_n = 0. \tag{3.2.21}$$

As before, this implies that the space $V_n = \text{Im } C_n = \text{Ker } D_n$ must be invariant under the action of the operators $\partial_x - U|_{\lambda-\lambda_n}$ and $\partial_t - V|_{\lambda=\lambda_n}$.

Now, the simplest way to satisfy these invariance conditions would be to simply write $W_n(\lambda; x, t)$ and $V_n(\lambda; x, t)$ as in (3.2.13). □

We will first point out that, though we have extended things such that the $\theta_\pm$ matrices may have zeroes, the factorization of $g(\lambda)$ must still have no zeroes. While it might be possible to relax this condition and allow g to be factored according to the Riemann-Hilbert problem with zeroes, we must take greater care. The ability to write the dressed connection (3.2.16) in terms of only $\theta_\pm$ requires us to utilize the invertibility of $g_\pm$ at all points within their respective domains of definition. If we were to allow $g_\pm$ to have zeroes, we would lose invertibility at those isolated points, and so it is possible for a singularity to be introduced even when we satisfy the subspace invariance conditions only because (3.2.16) might not hold at the zeroes of $g_\pm$. In the end, we would have to treat the isolated zeroes in $g_\pm$ separately from those in $\theta_\pm$ which would complicate the analysis significantly.

Even if we were to introduce this generalization, we would need to ask ourselves how it might actually aid us. We have already observed that $g_\pm$ plays no role in the form of the dressed Lax connection (3.2.16), so generalizing the allowed factorizations for g would only increase the freedom in Ψ^g, but since this freedom does not translate into freedom in the connection, we have not really achieved anything worthwhile in much the same way that refusing to fix an electromagnetic gauge might afford us a great deal of freedom in the 4-potential, but actually introduced no freedom in the physical fields.

To continue with this gauge analogy, we might as well fix a convenient "gauge" g for the problem, that is, one for which we can actually solve the Riemann-Hilbert factorization problem. Clearly, the identity represents a convenient choice for g since then the factorization problem is also trivial, $g_\pm = 1$. We might argue that would could have also chosen any constant matrix and then performed some kind of factorization on it, or even a diagonal matrix so the factorization problem is a series of scalar ones[15]. However, again, this would introduce no new freedoms into the Lax connection itself, and in fact, we shall recall that in defining the wave function Ψ, we stated that it would only be defined up to right multiplication by a constant matrix. What we have just proposed to find a simple solution to the Riemann-Hilbert problem for g is no more than picking some right multiplication.

At this point, we may notice that, in fact, the choice $g = 1$ does introduce freedom into the dressed connection since g plays a role in defining the Riemann-Hilbert problem which defines $\theta_\pm$. So, all of this about how g introduces *no* freedom into the connection is not quite correct. However, the Riemann-Hilbert problem which defines $\theta_\pm$ is one with zeroes. As we noted above, when we wish to generate Riemann-Hilbert factorizations with zeroes, we can construct them from some Riemann-Hilbert factorization without zeroes, but even the trivial factorization would yield non-trivial factorizations with zeroes. Thus, if we do choose $g = 1$, we are still capable of finding a wide variety of matrices $\theta_\pm$. In fact, by applying Theorem 7 repeatedly to the trivial factorization without zeroes, we are capable of obtaining

[15]The Riemann-Hilbert problem in the special case of scalars is a solved problem and formed a portion of Hilbert's PhD dissertation. For more information, see [12][2].

any number of factorizations to define $\theta_\pm$, so long as we are willing to define additional subspaces W_n and V_n to go with the additional zeroes we have added to the factorization.

So, though it is true that some implicit freedom is lost when we choose $g = 1$, we are still capable of obtaining countably[16] many dressed solutions just by choosing different numbers of zeroes to introduce into the factorization. In fact, though it is by no means proof, we shall present the following heuristic argument. Recall that we motivated the dressing transformations as some kind of permutation, or "rotation" within the integrability hierarchy developed in §2. Furthermore, recall that the hiegherarchy was a discrete set of countably many elementary flows.

If we then suppose that U, V and U^g, V^g are both connections with the same analytic structure, then we would expect them to sit somewhere in this discrete hierarchy of integrable systems. If we further suppose that they are distinct connections so that they do not occupy the same location in the hierarchy, then we may also require that the pair U^g, V^g has the form (3.2.16) for some[17] g. If we then smoothly transform g to g', in the same way we might smoothly deform the Hamiltonian when introducing a perturbation, we would obtain some new connection $U^{g'}, V^{g'}$. However, we have already stated that the hierarchy is a discrete object, so over the course of this transformation, we can only have passed through a discrete number of connections which are actually distinct in the hierarchy. If $U^{g'}, V^{g'}$ is to actually be distinct from U^g, V^g in the hierarchy, then there must be some point during this smooth transformation of g which transitioned the connection by a discrete step. So, it would seem to follow that $U^{g'}, V^{g'}$ must not really be distinct from U^g, V^g. Therefore, any connections we can obtain via dressing with some g, we should also be able to obtain using $g = 1$ since we can simply transform g to the identity.

Now, of course, that is far from a rigorous argument, and it certainly isn't completely airtight, but it does give us the sense that we should not end up missing out on anything by choosing $g = 1$, a choice which simplifies a great deal in the problem.

Before continuing on to demonstrate a pair of applications of the dressing technique to the non-linear sigma model and give an outline of the method's application in the case of the giant magnon, we must discuss the subspace condition (3.2.13). Theorem 6 states that specifying these subspaces is sufficient to define the solution to the factorization problem, and therefore, once g is decided, the dressing transformation. As noted in the discussion

[16] If we consider the locations of the poles, then there would appear to be uncountably many dressing solutions possible, however, we can always use a conformal mapping, like a Möbius transformation to move the poles around as we please. Therefore, the only real freedom is in the number of poles, which is a countable freedom.

[17] Note that we never proved that the group of dressing transformations can map us from a given connection to any other, but we seek here only to compare how much of the hierarchy is accessible with g arbitrary as opposed to $g = 1$.

following that Theorem, however, we never proved that this specification is also necessary.

We may further observe that in satisfying the invariance conditions (3.2.19) and (3.2.21), while it is sufficient to require (3.2.13), there is no reason to believe that this this is also a necessary condition. Though it is true that Ψ is the unique function, up to right multiplication by a constant matrix, which is in the kernel of the $(\partial - U)$ operators, we only needed for the image and kernel of the dressing matrices and their inverses to match and be invariant under the action of the $(\partial - U)$ operators about each pole. Even if this were not the case, we shall see in chapter §4 that there are some problems in which it may be beneficial to resort directly to the differential equations rather than work completely in terms of the images and kernels of various matrices about the poles.

§3.3 Dressing the Non-Linear Sigma Model

Perhaps the simplest model to dress is the non-linear sigma model presented as an example of an integrable field theory in §2.5. We recall that the connection was defined by

$$U = \frac{1}{\lambda - 1} J_x, \quad V = -\frac{1}{\lambda + 1} J_t, \tag{3.3.1}$$

where J_x and J_t are valued in some Lie algebra. For simplicity, we shall assume that they lie in a 2×2 representation of su(2) which is spanned by the Pauli matrices σ_j. Since, J_x and J_t are pure gauge, there exists a matrix $g(x,t)$ such that $J_x = g^{-1}\partial_x g$ and $J_t = g^{-1}\partial_t g$. We observe that the associated linear system is then equivalent to

$$\Psi \partial_x \Psi^{-1} = g^{-1}\partial_x g, \quad \Psi \partial_t \Psi^{-1} = g^{-1}\partial_t g, \tag{3.3.2}$$

when evaluated at $\lambda = 0$. Therefore,

$$g(x,t) = \Psi^{-1}(0; x, t). \tag{3.3.3}$$

A simple solution to the equations of motion are then given by

$$J_x = a\sigma_3, \ J_t = b\sigma_3, \ \Psi_0 = e^{\left[\left(\frac{ax}{\lambda-1} - \frac{bt}{\lambda+1}\right)\sigma_3\right]}, \ g_0 = e^{-(ax+bt)\sigma_3}, \tag{3.3.4}$$

for any constants a and b. To dress this simple solution, we write down the form of the dressing matrices with zeroes at μ_1 and λ_1 by Theorem 7,

$$\theta_+(\lambda) = \chi_0^{-1}\left(1 - \frac{\mu_1 - \lambda_1}{\lambda - \lambda_1} P\right), \quad \theta_-^{-1}(\lambda) = \left(1 - \frac{\lambda_1 - \mu_1}{\lambda - \mu_1} P\right)\chi_0. \tag{3.3.5}$$

where χ_0 is constant with respect to λ, and P is some projection. We may simply take P to be a rank one projection, which may therefore be parametrized by

$$P_i{}^j = \frac{w_i v^j}{w_k v^k}, \tag{3.3.6}$$

where w is some vector and v some covector, both of which may generally depend on x and t, but not on λ. The conditions (3.2.7) then inform us that w spans the image space W and that the vectors orthogonal to v span the space V. The invariance conditions (3.2.13) then imply that

$$w(x,t) = \Psi_0(\mu_1; x, t) w(0) \tag{3.3.7}$$

for some constant vector $w(0) \in W(0)$. If $v^\perp(0) \in V(0)$, then the action of $\Psi_0(\lambda_1; x, t)$ on the entire space allows us to write

$$v^\perp(x,t) = \Psi_0(\lambda_1; x, t) v^\perp(0). \tag{3.3.8}$$

Since Ψ_0 operates on the entire space by the invariance condition, our choice in $v^\perp$ is arbitrary. In this way, we also avoid any worries about the dimension of V, though since we have already fixed this example to be a two dimensional problem, the dimension of V is clearly one.

But now, we have things in terms of $v^\perp$ rather than v, so to find v in terms of $v^\perp$, we must search for a covector which is orthogonal to $v^\perp$. Thus, if we suppose $v = v(0)\Psi_0^{-1}(\lambda_1; x, t)$, then

$$v^k v_k^\perp = v(0)\Psi_0^{-1}(\lambda_1; x, t)\Psi_0(\lambda_1; x, t) v^\perp(0) = 0, \tag{3.3.9}$$

so our supposed form for v is the correct one. From here, we see that P has the form

$$P = \frac{\Psi_0(\mu_1) w(0) v(0) \Psi_0^{-1}(\lambda_1)}{v(0)\Psi_0^{-1}(\lambda_1)\Psi_0(\mu_1) w(0)}, \tag{3.3.10}$$

where we are free to choose $w(0)$ and $v(0)$. The dressed connection may then be reconstructed from the gauge field,

$$g(x,t) = \Psi^{-1}|_{\lambda=0} = \Psi_0^{-1}|_{\lambda=0}\ \theta_-^{-1}|_{\lambda=0} = g_0(x,t)\left(1 + \frac{\lambda_1 - \mu_1}{\mu_1} P\right)\chi_0. \tag{3.3.11}$$

The non-linear sigma model admits a simple dressing because the connection is described by a pure gauge. It is therefore sufficient to describe the dressing of the gauge factor g rather than write out the dressed connection in the form (3.2.16), though certainly we could have done so. Though this simplifies things, if we did not realize that the connection was given

by a pure gauge, things would not be much more complicated. Essentially, we would write down the dressed Ψ matrix and then solve the linear system for the dressed connection, a process which only requires us to take two derivatives of the dressed wave function.

The true reason that this model is easy to dress is that there are no additional symmetries that the connection is required to satisfy. All of the machinery of dressing transformations that we have developed till now has never said anything about satisfying additional symmetries. For example, in working through the dressing of the sigma model here, we could quite easily be accused of being overly pedantic about the distinction between vectors and covectors. After all, we are working in flat space, so we should be able to raise and lower indices at will.

In attempting to dress the giant gluon, a task performed in [10], the connection is again a pure gauge like we have here, but in that case, the gauge factor g is also required to be an element of SU(1,1), and so must satisfy $g^\dagger M g = M$, where $M = \mathrm{diag}(1, -1)$. The end effect on the form of the dressing matrix is that this M matrix acts like a metric on the space of vectors we must use to define our projection P. This immediately alters the form of P, and therefore also the dressing. The introduction of symmetries into the system put constraints on the form of the dressing matrix and it is these constraints which can alter the dressing technique from a completely algebraic and algorithmic procedure to a potentially untenable mess.

§3.4 Dressing the Giant Magnon

Here we will outline the dressing of the giant magnon, for greater detail the reader is referred to [13]. Strictly speaking, the giant magnon is a particular solution of the sigma model which satisfies certain symmetries. So, if we wish to build a new giant magnon solution from the original one, we must require also that the dressed solution satisfy the same symmetry conditions as the giant magnon, otherwise we have mapped ourselves to some other particle of the sigma model.

Additionally, this model uses a modified set of coordinates which are often used to simplify related problems. Rather than use the coordinates x and t, we utilize the coordinates $z = x + it$ and $\bar{z} = x - it$, which forms a perfectly good coordinate frame from any two dimensional system since we can invert it to return to the x, t coordinates. The derivatives transform accordingly and we will write $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. We redefine the connection so the associated linear system has the form

$$(\partial - U(\lambda; z, \bar{z}))\Psi(\lambda; z, \bar{z}) = 0, \quad (\bar{\partial} - V(\lambda; z, \bar{z}))\Psi(\lambda; z, \bar{z}) = 0. \tag{3.4.1}$$

For this model, just as in the previous section, the connection elements have the form

$$U = \frac{A}{1 - \lambda}, \quad V = \frac{B}{1 + \lambda}, \tag{3.4.2}$$

where both A and B are pure gauge so $A = g^{-1}\partial g$ and $B = g^{-1}\bar{\partial} g$, but now the gauge factor g is taken to be an element of $S^{N-1} = \mathrm{SO}(N)/\mathrm{SO}(N-1)$. To obtain a magnon solution, we must enforce the following additional symmetries:

$$\Psi^\dagger(\bar{\lambda})\Psi(\lambda) = 1, \quad \overline{\Psi(\bar{\lambda})} = \Psi(\lambda), \quad \Psi(\lambda) = \Psi(0)M\Psi(1/\lambda)M, \tag{3.4.3}$$

where $M = \mathrm{diag}(1, -1, \ldots, -1)$ is an $N \times N$ matrix.

With each of these conditions, we may write[18] $\Psi^g = \theta\Psi$ to show that the dressing matrix θ must satisfy the same symmetry conditions,

$$\theta^\dagger(\bar{\lambda})\theta(\lambda) = 1, \quad \overline{\theta(\bar{\lambda})} = \theta(\lambda), \quad \theta(\lambda) = \theta(0)\Psi(0)M\theta(1/\lambda)\Psi(0)M. \tag{3.4.4}$$

The first condition, unitarity, implies that whenever we have a pole at λ_1, the inverse $\theta^{-1}(\lambda)$ must have a pole at $\bar{\lambda}_1$. In the notation we have developed, this means that the factorization must have zeroes at $\mu_1 = \bar{\lambda}_1$ and λ_1. This first condition also implies that the projection P must additionally be Hermitian.

So, when we derive the form of P as we did in §3.3, we need only replace μ_1 by $\bar{\lambda}_1$, check that the form is Hermitian, and note that since our vectors are not in $\mathbb{C}^N$, the correct representation of the covector we called v in the previous section is the Hermitian conjugate of some vector. Therefore, if w and v are vectors independent of all variables,

$$P = \frac{\Psi(\bar{\lambda}_1)wv^\dagger\Psi^{-1}(\lambda_1)}{v^\dagger\Psi^{-1}(\lambda_1)\Psi(\bar{\lambda}_1)w}. \tag{3.4.5}$$

However, since this must be Hermitian, we require $w = v$.

If we now look at the remaining two constraints in (3.4.4), we see that if θ has a pole at λ_1, then it must also have one at $\bar{\lambda}_1 = 1/\lambda_1$, so both poles must lie on the unit circle and be conjugate to each other.

But now this implies that a single application of Theorem 7 will not do as it did with the sigma model under no further restrictions. We require at least two applications so that we can add two poles into θ. In general, this would result in four terms in the product, but we may choose the projectors to be orthogonal so that we are able to write[19]

$$\theta(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1}P + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}\overline{P}, \tag{3.4.6}$$

[18] Since we are choosing each of our Riemann-Hilbert factorizations without zeroes to be the trivial ones, we can choose any contour we please, and so if we require that $\theta_\pm$ be well-defined globally, then we can ignore the distinction between inside and outside the contour by requiring them to be global inverses. Thus, we suppress the $\pm$.

[19] Actually writing down the dressing matrix can at times be very difficult. Indeed, there is a four pole dressing of the giant magnon which requires a great deal of machinery to develop. The interested reader is referred to Theorem 4.2 and section 5 of [7].

where the orthogonality of the projectors in this form, along with forced satisfaction of (3.4.4) forces us to write $w^T w = 0$ and $\bar{w} = Mw$.

The remainder of the work in dressing the giant magnon reduces in large part to dealing with the vacuum solution. For our purposes, the development to this point is sufficient as it has demonstrated a salient point: the introduction of additional symmetry conditions can significantly complicate the process of writing down the dressing transformation.

§4 Dressing the Minimal Surface Problem

String theory is largely described as the search for two dimensional surfaces which have minimal surface area in some given manifold. The problem of interest in this paper comes from this surface minimization problem on the particular manifold known as AdS_5. The minimality condition may then be phrased in terms of a set of differential equations which, via a method known as the Pohlmeyer reduction, are transformed into the integrable system we seek to dress. The Pohlmeyer reduction is a lengthy procedure which relies heavily on connections the Lie structure of various symmetries that string solutions are forced to satisfy. As such, we shall not demonstrate the reduction here, but only motivate the problem and describe the results of the reduction. For a complete demonstration of the reduction, the reader is referred to either [4] or [9].

§4.1 Minimal Surfaces in AdS_5

With the framework developed to this point, we are finally in a position to begin applying the dressing technique to the problem of finding minimal surfaces in AdS_5 space, but first we must describe this problem. To begin, we will mention that AdS_n is a subspace of $\mathbb{R}^{2,n-1}$ given by the restriction

$$-(Y^{-1})^2 - (Y^0)^2 + (Y^1)^2 + \cdots + (Y^{n-1})^2 = -1. \tag{4.1.1}$$

In this space, we seek strings, known as worldsheets, which have minimal surface area. These world sheets come with a natural intrinsic coordinate system σ, τ where σ is spacelike and τ is timelike. We will, however, immediately abandon these coordinates in favor of the system $z, \bar{z}$ where $z = \sigma + i\tau$ and $\bar{z} = \sigma - i\tau$, just as we did in outlining the dressing of the giant magnon.

If we denote the scalar product in $\mathbb{R}^{2,n-1}$ space by the "$\cdot$" to avoid writing the metric repeatedly, we can recognize the object $\partial Y \cdot \bar{\partial} Y$ as the area "surface density" so that the

differential surface element is given by $\partial Y \cdot \bar{\partial} Y \mathrm{d}z\mathrm{d}\bar{z}$. To minimize the surface area, we need only minimize the action which is precisely the integral of this differential surface element,

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}(\partial Y \cdot \bar{\partial} Y - \Xi(Y \cdot Y + 1)), \tag{4.1.2}$$

where we have introduced the Lagrange multiplier Ξ so enforce the restriction (4.1.1) to AdS_5 space. This action functional S is known as the Polyakov action whose minimization yields the equations

$$\begin{aligned} \partial\bar{\partial}Y^k = (\partial Y \cdot \bar{\partial} Y)Y^k, \quad -1 \leq k \leq n-1, \\ Y \cdot Y = -1, \end{aligned} \tag{4.1.3}$$

which determine the potential string solutions in conjunction with the Virasoro constraint

$$\partial Y \cdot \partial Y = \bar{\partial} Y \cdot \bar{\partial} Y = 0, \tag{4.1.4}$$

which follows from a known symmetry satisfied by minimal surfaces.

A key piece of physical information contained in the string solution is the actual world-sheet area. To keep track of this quantity, we will define the function $\alpha(z, \bar{z})$ by

$$\partial Y \cdot \bar{\partial} Y = e^{2\alpha(z,\bar{z})}. \tag{4.1.5}$$

Our objective then, is to construct a system of differential equations for α. It is this task that the Pohlmeyer reduction accomplishes. Though we will not perform the reduction here, we specialize to AdS_5 so $n = 5$ and find that the result is the system

$$\begin{aligned} &2\partial\bar{\partial}\alpha + e^{-2\alpha}\cosh(2b) - e^{2\alpha} = 0, \\ &2\partial\bar{\partial}b - \sinh(2b)\left(e^{-2\alpha} + \frac{\partial\Lambda\bar{\partial}\Lambda}{\cosh^4(b)}\right) = 0, \\ &\bar{\partial}\left(\tanh^2(b)\partial\Lambda\right) + \partial\left(\tanh^2(b)\bar{\partial}\Lambda\right) = 0, \end{aligned} \tag{4.1.6}$$

where α and Λ are real valued functions and b is a pure imaginary function. So, the system (4.1.3) and (4.1.4) has been reduced to three function degrees of freedom as we would expect since there were three constraints in the six embedding coordinates Y^k in the original system.

Before proceeding, it is worthwhile to point out that in the special case where Λ is a constant and $b = 0$, only the first equation of (4.1.6) remains and we have the Cosh-Gordon equation whose solutions were found in terms of the Riemann theta functions by [9].

Now, we will also observe that the system (4.1.6) is generated by the Lagrangian density

$$\mathcal{L} = \partial\alpha\bar{\partial}\alpha + \partial b\bar{\partial}b + \tanh^2(b)\partial\Lambda\bar{\partial}\Lambda + \frac{1}{2}\left(e^{-2\alpha}\cosh(2b) + e^{2\alpha}\right). \tag{4.1.7}$$

I then follows that this system must have a Hamiltonian representation, and though we do not care to write it explicitly, this does motivate us to search for a Lax connection which generates the system. Fortunately, the underlying Lie structure exploited in the Pohlmeyer reduction also presents a straightforward construction for the connection.

The complete form of the Lax connection is shown in [4] and admits a simple representation in terms of the Lie structure, but for our purposes, only need to know that the connection has the form

$$\partial\Psi = A\Psi, \quad \bar{\partial}\Psi = \bar{A}\Psi, \tag{4.1.8}$$

where

$$\begin{aligned} A &= A_0 + \lambda A_1, \\ \bar{A} &= \bar{A}_0 + \frac{1}{\lambda}\bar{A}_1. \end{aligned} \tag{4.1.9}$$

We have used the bar in this instance only as notation to keep track of which connection element pairs with the $\bar{z}$ derivative. The matrix $\bar{A}$ need not be the conjugate of A.

Before proceeding, we will remark on the similarity between the form of the connection (4.1.9) and the form of the connection in the Sinh-Gordon model, (2.5.7). This should be expected since a special subset of solutions to our system have already been pointed out to satisfy the closely related Cosh-Gordon equation. The difference here is that we have additional constraints which will manifest themselves as additional symmetries when we try to dress solutions of this system.

For the interested reader, solutions to (4.1.6) have been found in a purely formal sense in [3]. These formal solutions rely on a path ordered integration of a series of matrices defined formally in terms of the Lie structure. The nature of this treatment is, however, such that explicit solutions are not possible and so we cannot actually use this approach to calculate the function α, thus leaving the important physical information contained in the worldsheet area outside of our reach.

§4.2 Vacuum Solution

When dressing a system, any vacuum solution will work as a basis for the dressing transformation, however, we do need to actually be able to write it down. We would therefore like to choose a simple solution. Recalling that the form of the zero curvature equation is

$$\partial\bar{A} - \bar{\partial}A - [A, \bar{A}] = 0, \tag{4.2.1}$$

one such simple solution might satisfy

$$\bar{\partial}A = 0, \quad \partial\bar{A} = 0, \quad [A, \bar{A}] = 0. \tag{4.2.2}$$

If $G(z)$ and $\bar{G}(\bar{z})$ are arbitrary holomorphic functions[20], then one ansatz satisfying (4.2.2) is

$$A = \lambda \partial G M, \quad \bar{A} = \frac{1}{\lambda} \bar{\partial} \bar{G} (-M)^3, \tag{4.2.3}$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.2.4}$$

We will further note that $M^4 = -1$, so M is a $\mathbb{Z}_8$ representation. This will allow us to exponentiate and invert polynomials in M through simple algebraic means.

Writing the associated linear system

$$(\partial - A)\Psi = 0, \quad (\bar{\partial} - \bar{A})\Psi = 0, \tag{4.2.5}$$

we may separate variables by writing $\Psi = \Psi_z \Psi_{\bar{z}}$ and integrate the pair of now ordinary differential equations for $\Psi_z(z)$ and $\Psi_{\bar{z}}(\bar{z})$. The expressions that result for the separation are lengthy and not particularly enlightening or important. We will, however, record the resulting form of Ψ since it is important to observe that this is also a polynomial in M,

$$\begin{aligned} \Psi(z, \bar{z}) = &\cosh(G_+)\cos(G_-) + \frac{1}{\sqrt{2}} \left(\cosh(G_+)\sin(G_-) + \sinh(G_+)\cos(G_-)\right) M + \\ &\sinh(G_+)\sin(G_-) M^2 + \frac{1}{\sqrt{2}} \left(\cosh(G_+)\sin(G_-) - \sinh(G_+)\cos(G_-)\right) M^3, \end{aligned} \tag{4.2.6}$$

where we have defined the function

$$G_\pm(\lambda; z, \bar{z}) = \frac{\lambda G(z)}{\sqrt{2}} \pm \frac{\bar{G}(\bar{z})}{\lambda \sqrt{2}}. \tag{4.2.7}$$

It is this solution that we will take as our vacuum soliton when attempting to dress the minimal surface problem via the techniques developed in §3.

§4.3 Symmetry Conditions

As mentioned in §4.1, the similarity between the Cosh-Gordon equation and the Sinh-Gordon equations would lead us to believe that our will also have some symmetries to impose the additional constraints represented by the additional equations in the system (4.1.6), much

[20] We again employ this bar notation to indicate that $\bar{G}$ goes with $\bar{z}$. It need not be the conjugate of G.

like we required additional symmetries in the case of the giant magnon in §3.4 to restrict the model to a certain type of solution identified as a magnon.

In our case, these symmetries can be described by the two equalities

$$\Psi(\lambda) = C_1 \left(\Psi^{-1}(i\lambda)\right) C_1, \quad \Psi(\lambda) = C_2 \left(\Psi^{-1}(1/\bar{\lambda})\right)^\dagger C_2, \tag{4.3.1}$$

where the matrices C_1 and C_2 are the Hermitian $\mathbb{Z}_2$ representations

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{-1+i}{\sqrt{2}} \\ 0 & 0 & \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} & 0 & 0 \\ \frac{-1-i}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{4.3.2}$$

If we now do exactly as we did with the giant magnon and require the dressed solution $\theta\Psi$ also satisfy the symmetry conditions (4.3.1), we find that the dressing matrix θ must also satisfy the same relations,

$$\theta(\lambda) = C_1 \left(\theta^{-1}(i\lambda)\right)^T C_1, \quad \theta(\lambda) = C_2 \left(\theta^{-1}(1/\bar{\lambda})\right)^\dagger C_2. \tag{4.3.3}$$

Repeated application of the symmetries (4.3.3) show that they return to identity after four applications, each giving a kind of Z_4 symmetry. Therefore, if we have a pole at $\lambda = \lambda_1$ in $\theta(\lambda)$, we are also forced to have three additional poles in θ for a total of four along with four poles in $\theta^{-1}(\lambda)$. Applying the symmetries (4.3.3) repeatedly to find the locations that the pole λ_1 is mapped to, we find the following pole locations:

$$\begin{aligned} \theta(\lambda): &\quad \lambda_1, \quad -i/\bar{\lambda}_1, \quad -\lambda_1, \quad i/\bar{\lambda}_1, \\ \theta^{-1}(\lambda): &\quad 1/\bar{\lambda}_1, \quad -i\lambda_1, \quad -1/\bar{\lambda}_1, \quad i\lambda_1. \end{aligned} \tag{4.3.4}$$

One way to write θ would then be to require each of these poles to be simple, then one was to construct θ would be to write

$$\begin{aligned} \theta =& \chi_0 + i\frac{\lambda+\lambda_1}{\lambda-\lambda_1} Q + i\frac{\lambda - i/\lambda_1^*}{\lambda + i/\lambda_1^*} C_2 C_1 Q^* C_1 C_2 \\ &+ i\frac{\lambda-\lambda_1}{\lambda+\lambda_1} C_2 C_1 C_2^T C_1^T Q C_1^T C_2^T C_1 C_2 \\ &+ i\frac{\lambda + i/\lambda_1^*}{\lambda - i/\lambda_1^*} C_1^T C_2^T Q^* C_2^T C_1^T, \end{aligned} \tag{4.3.5}$$

for some matrices χ_0 and Q which are constant with respect to λ. Note that we have used the * notation to indicate the complex conjugate here to make conjugation more transparent

among the fraction bars. We may then use the relations (4.3.1) to compute the form of θ^{-1} in terms of the χ_0 and Q matrices. Doing so, we obtain

$$\begin{aligned}\theta^{-1} =& C_2\chi_0^\dagger C_2 + i\frac{\lambda+1/\lambda_1^*}{\lambda-1/\lambda_1^*}C_2Q^\dagger C_2 + i\frac{\lambda-i\lambda_1}{\lambda+i\lambda_1}C_1Q^TC_1 \\ &+ i\frac{\lambda-1/\lambda_1^*}{\lambda+1/\lambda_1^*}C_1C_2^TC_1^TQ^\dagger C_1^TC_2^TC_1 \\ &+ i\frac{\lambda+i\lambda_1}{\lambda-i\lambda_1}C_2C_1^TC_2^TC^TC_2^TC_1^TC_2,\end{aligned} \tag{4.3.6}$$

though for both of the relations in (4.3.1) to result in this form, we must also require

$$C_2C_1\chi_0^*C_1C_2 = \chi_0. \tag{4.3.7}$$

These forms are determined entirely from the symmetry conditions, therefore, (4.3.5) and (4.3.6) may not actually be inverses. Indeed, it is entirely possible that they are not even invertible. Imposing the condition $\theta(\lambda)\theta^{-1}(\lambda) = 1$ for all λ is generally non-trivial system of quadratic equations and hence is a difficult condition to impose.

§4.4 Imposing the Invertibility Condition

We must require that the forms (4.3.5) and (4.3.6) for θ and θ^{-1} are actually inverses of each other for all λ. However, since the complete information of a complex function is contained at its poles, we need only require that they are inverses of each other about the poles to guarantee invertibility for all λ.

In imposing these conditions, however, it would be sufficient to impose the condition only about the pole at λ_1 since the symmetry conditions (4.3.3). The symmetry conditions map us from pole to pole, and so if the invertibility condition is satisfied at one of the poles, then it must also be satisfied at all of the poles, and we might as well choose λ_1 to be that pole since it generates the other poles in the function. If we were to write out the invertibility condition about each of the poles, this symmetry is immediately evident as the resulting equations are easily seen to be completely equivalent.

So, while this realization that we only need to require invertibility about λ_1 simplifies the problem at hand significantly, we are still left with a system of quadratic equations in the matrix elements of Q and χ_0. Expanding the forms (4.3.5) and (4.3.6) explicitly about their poles, we require that the non-polar parts of θ and θ^{-1} to product to unity and that the residue about λ_1 be zero, that is, $2i\lambda_1 Q\theta^{-1}(\lambda_1) = 0$. The equations to be solved may

then be written explicitly in the form

$$1 = \left(\chi_0 + iQ + iC_2C_1Q^*C_1C_2 + iC_2C_1C_2^TC_1^TQC_1^TC_2^TC_1C_2 + iC_1^TC_2^TQ^*C_2^TC_1^T\right)\cdot \left(C_2\chi_0^\dagger C_2 + iC_2Q^\dagger C_2 + iC_1Q^TC_1 + iC_1C_2^TC_1^TQ^\dagger C_1^TC_2^TC_1 + iC_2C_1^TC_2^TQ^TC_2^TC_1^TC_2\right), \tag{4.4.1}$$

$$\begin{aligned} 0 =& Q\left(C_2\chi_0^\dagger C_2 + iC_2Q^\dagger C_2 + iC_1Q^TC_1 + iC_1C_2^TC_1^TQ^\dagger C_1^TC_2^TC_1 + \right. \\ & iC_2C_1^TC_2^TQ^TC_2^TC_1^TC_2 + \frac{2i}{|\lambda_1|^2-1}C_2Q^\dagger C_2 + \frac{2}{1+i}C_1Q^TC_1 - \\ & \left.\frac{2i}{|\lambda_1|^2+1}C_1C_2^TC_1^TQ^\dagger C_1^TC_2^TC_1 - \frac{2}{1-i}C_2C_1^TC_2^TQ^TC_2^TC_1^TC_2\right). \end{aligned} \tag{4.4.2}$$

In particularly, we note that only the square modulus of λ_1 appears. This seems to imply that, whatever these constraints ask of our variables, we would expect at least an entire circle's worth of freedom in the solution, even if all other variables are completely determined. Secondly, we will point our that if we view these equations strictly as a set of quadratic equations in the matrix elements of Q and χ_0, there *does* exist a solution. Namely, this solution is $\chi_0 = 1$ and $Q = 0$. This solution is, of course, worthless to us since it would imply that our dressing is a trivial one, but the existence of a solution at all can give us a starting place for solving the system as though it were strictly a quadratic system.

The structure of this system has been examined at length, and a number of methods have been developed in an attempt to find non-trivial solutions to the system. We will outline some of these methods to give the reader a better sense of the system and the challenges presented in finding a solution.

If we expand the matrix elements of Q and χ_0 in terms of their real and imaginary parts, we find a total of 64 variables in total, not including the spectral parameter. The condition (4.3.7) is a completely linear one since C_1 and C_2 are known constant matrices. Solving this equation eliminates precisely half of the variables present in χ_0, leaving us with 48 variables to satisfy the quadratic system given by (4.4.1) and (4.4.2). Separating these equations into real and imaginary parts, there are 64 equations. Furthermore, 58 of these equations are linearly independent.

Now, the initial reaction might be that this system cannot have a solution because there are more equations than unknowns and the equations are not dependent on each other. But, we know with certainty that this intuition is false because we have already mentioned that $Q = 0, \chi_0 = 1$ represents a solution to the quadratic system. To gain an idea how this could be the case, we must remember that this intuition comes from the study of linear equations. The same rules do not apply for non-linear equations. As an example, it is a simple exercise

to construct three ellipses centered at the same point which all pass through some chosen point. To show this mathematically, recall that ellipses have the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.4.3}$$

for some constants a and b. If we only wish that this conic section pass through (x_0, y_0), then we may write $b^2x_0^2 + a^2(y_0^2 - b^2) = 0$ so

$$a^2 = \frac{b^2x_0^2}{b^2 - y_0^2}, \tag{4.4.4}$$

which generates a solution so long as the right hand side is positive. But this is merely the constraint $b > \pm y_0$. Even excluding the negative branch because the negatives will not generate new ellipses for us, we find that, in fact, there are infinitely many distinct ellipses which are centered at the origin and pass though this arbitrary point, at least so long as $(x_0, y_0) \neq (0, 0)$ since there is only one degenerate ellipse.

With this in mind, we will abandon the matrix formalism to study the invertibility condition as a quadratic system. We will then write the 58 equations in 48 variables in the form

$$\mathbf{x}^T J_i \mathbf{x} = g_i, \quad 1 \leq i \leq 58, \tag{4.4.5}$$

for some symmetric 48×48 matrices J_i and real scalars g_i. The vector $\mathbf{x}$ contains the 48 variables of the system. These J_i matrices are relatively sparse and have a very advantageous feature: the diagonal of every J_i is zero. This implies that nowhere in the entire quadratic system do we find a single variable multiplying itself, each variable only multiplies other variables.

Still, this is a very large and very non-linear system, large enough that it would be difficult to treat by hand to say the least. One way to try and handle this problem would be to change the problem into a different one. The constraint (4.4.5) written in index notation as $x^k x_j J_k{}^j{}_i = g_i$ makes it clear that if we denote $\mathbf{x} \otimes \mathbf{x} = X$, then (4.4.5) may be written as

$$\mathrm{Tr}(J_i X) = g_i. \tag{4.4.6}$$

Since X is some symmetric matrix, we are able to expand it in terms of some basis E_i of symmetric matrices to write

$$X = \sum y_j E_j, \tag{4.4.7}$$

which allows us to write (4.4.6) as $\sum y_j \mathrm{Tr}(J_i E_j) = g_i$. If we define the 58×1176 matrix $T_{ij} = \mathrm{Tr}(J_i E_j)$, we may write

$$T\mathbf{y} = \mathbf{g}, \tag{4.4.8}$$

which is now a linear problem to determine a symmetric matrix X. In general, however, this matrix X may not be rank one and thus may not be equal to $\mathbf{x} \otimes \mathbf{x}$ as we desire. Since X is symmetric, it is enough to require that it be rank one since any rank one matrix which is not a multiple of $\mathbf{x} \otimes \mathbf{x}$ will also not be symmetric. Thus, we have translated the problem of finding solutions to a quadratic system into the problem of requiring a matrix known in terms of the parameters $\mathbf{y}$ to be rank one. If we are able to require X to be rank one, then we need only read off the square roots of the diagonal to find the solution $\mathbf{x}$.

There is certainly more than one way that rank one matrices are naturally defined, and thus there is more than one way we might go about trying to require that X is rank one. The first option to try would be to require at each of the columns are scalar multiples of the first column. That is, if $C(X, j)$ is the j^{th} column of X, then we wish to require that $C(X, 1) = \alpha_j C(X, j)$ for each j and $\alpha_1 = 1$. However, this creates a set of 48 vector equations. Each of these vector equations consists of 48 equations coupled, at the very least, but the α_j corresponding to that column. Furthermore, the vector equations are coupled because the dependency on the variables $\mathbf{y}$ is distributed over all of X, not arranged column by column. Therefore, this method has simply taken our quadratic system and given us a new one with both more variables and more equations.

A more promising method would be to require that X has precisely one non-zero eigenvalue. The characterization of rank one matrices as having exactly one non-zero eigenvalue follows from the following Theorem.

Theorem 9. Let M be any symmetric $n \times n$ matrix with eigenvalues λ_j and associated eigenvectors $\mathbf{v}_j$ with unit norm. Then we may write

$$M = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \otimes \mathbf{v}_j. \tag{4.4.9}$$

The proof of this Theorem follows immediately from the diagonalizability of symmetric matrices and is not of any particular interest to us. The implication of this Theorem, however, is that if a symmetric matrix has only one non-zero eigenvalue, then it must be rank one. The requirement that the $\mathbf{v}_j$ have unit norm is not of great concern to use since we can always absorb the scaling by λ_j into the definition of the eigenvectors.

So, we must determine whether we can find the eigenvalues of X. If we can, then we need only set $48 - 1$ of them to zero. We could interpret the choice in which eigenvalue to leave as nonzero as the source of branch cuts what we would typically expect in solutions to a quadratic system. The issue, of course, is that X is a 48×48 matrix, and since the T matrix is quite sparse, X is not too far off from being a completely general symmetric matrix, containing nearly all of the 1176 variables in $\mathbf{y}$. This implies that finding the eigenvalues

of X would require us to not only compute the determinant, but then also solve an almost completely general 48^{th} order polynomial.

To work around this issue, we might be to force all but one of the eigenvalues to zero by writing the characteristic polynomial in the form $\lambda^{48} + a_{47}\lambda^{47} + \cdots + a_0 = 0$. But if this polynomial is to have only a single non-zero root, all of the coefficients must be zero with the exception of a_{47}. The values of a_{47} would then be precisely the value of the only non-zero eigenvalue. These coefficients are generally functions of matrix elements of X, and therefore of the variables $\mathbf{y}$, though they would also generally be polynomials in the $\mathbf{y}$. The polynomial nature of these constraints, however, should be easier to deal with than the polynomial nature of the original quadratic problem simply because this problem contains only 47 equations but over 1000 variables. This variable supersaturation gives us hope to actually find a solution to the system, assuming we can actually write the coefficients as functions of the $\mathbf{y}$.

Here, however, we reach the final, insurmountable, problem with the quadratic system: shear size. There is an algorithm known as the Leverrier-Faddeev algorithm which computes the coefficients of the characteristic polynomial, described in Appendix 2. This algorithm allows us to write the condition that the coefficients are zero in the form

$$\mathrm{Tr}(X^j) = (\mathrm{Tr}X)^j, \quad 2 \leq j \leq 48. \tag{4.4.10}$$

Thus, so long as we can compute the first 48 powers of X and the corresponding traces, we are able to write down the system we must satisfy. The shear size of the system, however, makes this impossible to do symbolically with modern computers. The expressions involved become so lengthy that even we had an unbounded quantity of memory at our disposal, rough estimates place the runtime of the Leverrier-Faddeev algorithm at 37,000 years, or about a third of a rotation for the Milky Way. If we had the foresight to begin running this computation during the last glacial maximum, we would only now be receiving an output.

Needless to say, the shear size of this quadratic system represents the biggest challenge in finding solutions, though there may yet be other possibilities for future attempts, which we shall discuss in §5.

§4.5 Satisfying the Invariance Condition

Even were we capable of satisfying the invertibility condition, we must also satisfy the invariance conditions (3.2.13). Since the form we have found for the dressing matrix is in terms of generic matrices, it does not lend itself to statements about images and kernels. Therefore, we would need to attempt to force whatever solution we found for the invertibility condition to satisfy the differential equations (3.2.19) and (3.2.21) which are used to deduce the invariance conditions (3.2.13).

Since the invertibility condition is a complex quadratic system, we would also expect its solutions to be rather complicated. This doesn't give us much hope for satisfying a set of differential equations in the free parameters left over from the invertibility condition.

Another approach would be to try and satisfy this differential constraint before the invariance condition. This way, even if we are forced to find some solution to the invertibility condition with absolutely no free parameters, that would be okay since we have already satisfied the other constraints placed on the dressing matrix, namely the invariance condition and the symmetry conditions which make a mess of this problem in the first place.

A development of the theory of two dimensional partial differential matrix valued equations is presented in Appendix 3. A development of cyclic exponential theory is presented in Appendix 4. Without this theory of cyclic exponentials, the results of Appendix 3 are not particularly useful since solutions necessarily involve matrix exponentials.

The next thing to observe is that, although we have eight pole locations in our dressing matrix and its inverse, we are able to freely map between these pole locations with the symmetry conditions. This is how we obtained the forms (4.3.5) and (4.3.6) in the first place. Therefore, it would be sufficient to impose the invariance condition at a single pole.

So, we must first give the full polar expansion of the forms (4.3.5) and (4.3.6). These forms are:

$$\begin{aligned}\theta(\lambda) &= \chi_0 + iQ + iC_2C_1Q^*C_1C_2 + iC_2C_1C_2^TC_1^TQC_1^TC_2^TC_1C_2 + iC_1^TC_2^TQ^*C_2^TC_1^T \\ &\quad + \frac{2i\lambda_1}{\lambda-\lambda_1}Q + \frac{2/\lambda_1^*}{\lambda+i/\lambda_1^*}C_2C_1Q^*C_1C_2 - \frac{2i\lambda_1}{\lambda+\lambda_1}C_2C_1C_2^TC_1^TQC_1^TC_2^TC_1C_2 \\ &\quad - \frac{2/\lambda_1^*}{\lambda-i/\lambda_1^*}C_1^TC_2^TQ^*C_2^TC_1^T,\end{aligned} \tag{4.5.1}$$

$$\begin{aligned}\theta^{-1}(\lambda) &= C_2\chi_0^\dagger C_2 + iC_2Q^\dagger C_2 + iC_1Q^TC_1 + iC_1C_2^TC_1^TQ^\dagger C_1^TC_2^TC_1 + iC_2C_1^TC_2^TQ^TC_2^TC_1^TC_2 \\ &\quad + \frac{2i/\lambda_1^*}{\lambda-1/\lambda_1^*}C_2Q^\dagger C_2 + \frac{2\lambda_1}{\lambda+i\lambda_1}C_1Q^TC_1 - \frac{2i/\lambda_1^*}{\lambda+1/\lambda_1^*}C_1C_2^TC_1^TQ^\dagger C_1^TC_2^TC_1 \\ &\quad - \frac{2\lambda_1}{\lambda-i\lambda_1}C_2C_1^TC_2^TQ^TC_2^TC_1^TC_2.\end{aligned} \tag{4.5.2}$$

A convenient pole location to impose the invariance condition is then $1/\lambda_1^*$ so the invariance condition reads

$$\theta|_{\mathcal{O}(1)}\left(\partial - A|_{\lambda=\frac{1}{\lambda_1^*}}\right)C_2Q^\dagger C_2 = 0, \quad \theta|_{\mathcal{O}(1)}\left(\bar\partial - \bar A|_{\lambda=\frac{1}{\lambda_1^*}}\right)C_2Q^\dagger C_2 = 0, \tag{4.5.3}$$

where

$$\theta|_{\mathcal{O}(1)} = \chi_0 + iQ + iC_2C_1Q^*C_1C_2 + iC_2C_1C_2^TC_1^TQC_1^TC_2^TC_1C_2 + iC_1^TC_2^TQ^*C_2^TC_1^T. \tag{4.5.4}$$

As discussed in the proof of Theorem 8, we must have Ker $\theta|_{\mathcal{O}(1)} = \text{Im}\left(C_2Q^\dagger C_2\right)$ along with $\text{Im}\left(C_2Q^\dagger C_2\right) = \text{Im}\left(\nabla - \mathbf{A}\right)C_2Q^\dagger C_2$ where we have written $\nabla = \hat{z}\partial + \hat{\bar{z}}\bar{\partial}$ and $\mathbf{A} = A\hat{z} + \bar{A}\hat{\bar{z}}$ evaluated at $1/\lambda_1^*$.

If we then define $W(z,\bar{z})$ to be some matrix whose image is equal to that of $C_2Q^\dagger C_2$, then the constraints discussed to this point may be written as

$$\nabla\left(C_2Q^\dagger C_2\right) - \mathbf{A}C_2Q^\dagger C_2 = \mathbf{W}, \tag{4.5.5}$$

where $\mathbf{W}$ is the vector of matrices such that $W = W_1 = W_2$, in addition to the constraints Im $C_2Q^\dagger C_2 =$ Im W and Ker $\theta|_{\mathcal{O}(1)} =$ Im W. But now (4.5.5) is of the form (A3.1) with $X = C_2Q^\dagger C_2$, $\mathbf{P} = -\mathbf{A}$, and $\mathbf{W}$ as itself. We may then write the solution for Q by the work in Appendix 3 so long as $\bar{\partial}A = \partial\bar{A}$ at $1/\lambda_1^*$ and $AW + \partial W = \bar{A}W + \bar{\partial}W$. The first of these conditions necessarily holds because A is not a function of $\bar{z}$ and $\bar{A}$ is not a function of z. The second condition is not quite so simple and represents a restriction on the allowable functional forms of W.

To present the result explicitly, Q is given by

$$Q = C_2e^{F^\dagger}(K^\dagger + C)C_2 \tag{4.5.6}$$

where C is an integration constant matrix, F is given by

$$F(z,\bar{z}) = \frac{1}{\lambda_1^*}G(z)M - \lambda_1^*\bar{G}(\bar{z})M^3, \tag{4.5.7}$$

and K is given by

$$\begin{aligned} K(z,\bar{z}) = &\int^z e^{-F(z',\bar{z})}W(z',\bar{z})\mathrm{d}z' + \int^{\bar{z}} e^{-F(z,\bar{z}')}W(z,\bar{z}')\mathrm{d}\bar{z}' \\ &- \int^{\bar{z}}\bar{\partial}'\int^z e^{-F(z',\bar{z}')}W\mathrm{d}z'\mathrm{d}\bar{z}'. \end{aligned} \tag{4.5.8}$$

The expression (4.5.7) admits an expression in terms of the cyclic exponential theory described in Appendix 4 so we may write

$$e^F = \left(\sum_{j=0}^{7} M^j\mu_j^8\left(\frac{1}{\lambda_1^*}G(z)\right)\right)\left(\sum_{j=0}^{7} M^{3j}\mu_j^8\left(-\lambda_1^*\bar{G}(\bar{z})\right)\right) \tag{4.5.9}$$

The work continuing from this point is still in progress. We shall discuss planned forays deeper into these differential equations in §5 along with other possible methods for producing a satisfactory dressing matrix.

§5 Future Directions

Though a great deal has been developed to constrain the form of the dressing matrix, an explicit form for the dressing has, as of yet, not been produced. There are, however approaches which have not yet been ruled out as impractical like those described in §4.4.

The most immediate and promising approach appears to what we have just described in §4.5 to satisfy the invariance condition. Once the allowable forms for W are determined, then imposing the Ker $\theta|_{\mathcal{O}(1)}$ = Im W condition becomes a completely linear problem in both the elements of χ_0 and the elements of W. Similarly, Im $C_2Q^\dagger C_2$ = Im W is a linear problem in the matrix elements of W.

If we are able to satisfy these conditions, then we have also progressed towards a solution for the invertibility condition. We can see this if we recall that invertibility requires that θ and θ^{-1} are actually inverses at one of the poles. If we take this pole to be $1/\lambda_1^*$ rather than λ_1 as we did in §4.4, then we will notice that again we have two equations to satisfy. The first is the condition that the non-polar parts product to unity. This equation will not change, but we will find that the second equation is already satisfied since the condition Ker $\theta|_{\mathcal{O}(1)}$ = Im W = Im $C_2Q^\dagger C_2$ implies that the residue of the product at $1/\lambda_1$ vanishes. Thus, we would be left with exactly one quadratic equation: the invertibility of the zeroth order terms of θ and θ^{-1}.

It is possible that this reduction in the size of the quadratic system will make one of the methods described in §4.4 feasible to use in practice. Another possibility for finding solutions to the quadratic system lies in the nice structure of the system and fact that there is a known solution to the system, $Q = 0$, $\chi_0 = 1$, though this solution is useless to us. Since all of the matrices J_i defined in (4.4.5) have only zeros along the diagonal, no variable in the system ever multiplies itself. Since nearly all of the g_i are zero, we have the opportunity to attempt a discrete version of perturbing the system about the known solution. Essentially, we would start with the known solution and then select a position in the $\mathbf{x}$ vector to be free. It would then be a simple matter to check if this perturbed vector still satisfies (4.4.5). The fact that $Q = 0$ is part of the known solution gives hope that this method would be successful since most of the variables in $\mathbf{x}$ will be zero. When we free one of the variables, there is then a good chance that every occurrence of the freed variable will appear in conjunction with a variable set to zero, thus satisfying the system trivially.

With the quadratic system significantly reduced in size, there is also the possibility of performing a decomposition of the system into a basis of polynomials known as the Gröbner basis. This basis is a particularly convenient basis for polynomials which has the nice property of being applicable even in multivariate systems. There are several algorithms for computing a decomposition to the Gröbner basis, though they are too slow to handle a system with the size and complexity of the one presented in §4.4. With the reduction in system size, however, this method may become a feasible option for treating the system.

§6 Conclusions

The theory of integrable systems is capable of bringing us a long was from the Hamiltonian systems which originated the study of integrable systems. The theory also presents powerful methods for generating new solutions to integrable field theories. Indeed, this method is rather simple to execute given no other constraints, as we saw when dressing the non-linear sigma model.

Rather, it is the additional constraints added beyond those inherent in the model specified by the pole structure of the field theory which cause the problem of writing down a dressing matrix a complex one. Even the dressing of the giant magnon with its relatively minor symmetry conditions increases in complexity over the plain non-linear sigma model almost immediately, if only from the requirement that there must be an even number of poles. The four pole dressing matrix for the giant magnon takes even more to write down still.

So, when we move to the problem of dressing the Pohlmeyer reduced AdS_5 minimal surface problem with the symmetry conditions (4.3.1), it should be no surprise that the problem of writing down a dressing matrix becomes even more complex. Therefore, although we have not yet been able to write down the form of the dressing matrix, the study of the invertibility and invariance conditions, which have been presented here, have led to a great plethora of knowledge about the constraints that the dressing matrix must satisfy.

The most recent successes in writing the form of the Q matrix which satisfies the invariance condition and the possibility of satisfying half of the quadratic equations arising from the invertibility condition via linear means, leave us hope that we may yet write down an explicit dressing matrix and generate new and explicit solutions to the minimal surface problem in AdS_5 space.

A1 The Cayley-Hamilton Theorem

We present here the Cayley-Hamilton Theorem and use it to show that the powers of a diagonal matrix with distinct nonzero elements form a basis for diagonal matrices.

Theorem 10. Let A be an $n \times n$ matrix with characteristic equation

$$p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + (-1)^n \det(A) = 0, \tag{A1.1}$$

then A satisfies its own characteristic polynomial. That is,

$$p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + (-1)\det(A)I_n = 0. \tag{A1.2}$$

Though this is without a doubt a beautiful result, we will not provide a proof of it here. Instead, we will simply apply this result to show that the first n powers of a diagonal matrix A with distinct, non-zero, elements form a basis in the space of diagonal matrices.

Theorem 11. If A is an $n \times n$ diagonal matrix whose elements are distinct and non-zero, then the collection $\mathcal{B} = \{A, A^2, \ldots, A^n\}$ forms a basis for the space of $n \times n$ diagonal matrices.

Proof. We begin by observing that the space of $n \times n$ diagonal matrices has dimension n and will therefore require an n element basis to span. Therefore, we need only show that $\mathcal{B}$ is a linearly independent set. To this end, it would be sufficient to show that the characteristic polynomial $p(X)$ is the minimal polynomial of A since then A^n will be the smallest power of A which is a polynomial in the lower powers of A.

If we use the Cayley-Hamilton theorem to write characteristic polynomial of A in the form $p(X) = X^n + c_{n-1}X^{n-1} + \cdots + c_1 X + (-1)\det(A)I_n$, we can use the fact that A is diagonal to factor p into the form

$$p(X) = \prod_{j=1}^{n}(X - a_j I_n), \tag{A1.3}$$

where a_j is the j^{th} element along the diagonal of A and I_n is the $n \times n$ identity matrix. That is, the $a_j I_n$ are the "roots" of A, since this factorization causes $p(A) = 0$ as it must, and each factor $(A - a_j I_n)$ has zero determinant.

If we suppose towards a contradiction that the minimal polynomial of A is, say, $f(X)$, then in much the same way we may write

$$f(X) = \prod_{j=1}^{m}(X - d_j I_n), \tag{A1.4}$$

for some $m < n$. We might argue that this is not the general form of a degree m polynomial since we might wish to put a general diagonal matrix in the place of $d_j I_n$, however, this would not result in a polynomial with complex coefficients, but rather a polynomial with diagonal matrices as coefficients. Therefore, the matrices in the place of $d_j I_n$ must indeed be diagonal matrices with constant value along the diagonal.

But now for $f(A) = 0$ to hold, it must be the case that the diagonal of the product (A1.4) must be zero. Since all of the matrices in the product are diagonal, this in turn implies that, for each diagonal position j, there must be at least one factor, say with index k, such that $(A - d_j I_n)$ has a zero in the j^{th} diagonal position.

Now, since the diagonal elements of A are assumed to be distinct and non-zero, we observe that at most one diagonal element of $(A - d_k I_n)$ can be zero for a given k. Thus, (A1.4) can have at most $m < n$ diagonal entries equal to zero, contradicting the assumption that f is the minimal polynomial of A, indeed A cannot even satisfy $f(A) = 0$. It then follows that $p(X)$ is the minimal polynomial of A and the result follows. □

A2 The Leverrier-Faddeev Algorithm

The Leverrier-Faddeev Algorithm is a $\mathcal{O}(n^4)$ time complexity algorithm to compute the coefficients of the characteristic polynomial, $p(\lambda)$ of an $n \times n$ matrix A. The algorithm exploits Newton's formulae for the coefficients and rewrites them in a more computationally efficient manner.

If the characteristic polynomial of A has the form

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n, \tag{A2.1}$$

and we define $s_j = \mathrm{Tr}(A^j)$, then Newton's formulae reads

$$\begin{aligned} s_1 &= -c_1, \\ s_2 + c_1 s_1 &= -2c_2, \\ &\vdots \\ s_n + c_1 s_{n-1} + c_2 s_{n-2} + \ldots + c_{n-1} s_1 &= -n c_n. \end{aligned} \tag{A2.2}$$

We can reobtain the same sequence by introducing a new matrix Y_j and instead performing a series of matrix computations. The algorithm is reformulated as follows,

$$\begin{aligned} &Y_0 = A, \\ &Y_1 = AY_0 + c_1 A, && \text{where} \quad c_1 = -\mathrm{Tr}(Y_0), \\ &Y_2 = AY_1 + c_2 A, && \qquad\quad c_2 = -\tfrac{1}{2}\mathrm{Tr}(Y_1), \\ &\quad\vdots && \qquad\quad \vdots \\ &Y_n = AY_{n-1} + c_n A, && \qquad\quad c_n = -\tfrac{1}{n}\mathrm{Tr}(Y_{n-1}). \end{aligned} \tag{A2.3}$$

The interested reader is referred to either [8] or [6] for a more detailed description of the method and some of its applications.

A3 First Order Linear Matrix-Valued PDEs

In this section, we develop the theory of first order linear partial differential equations in two dimensions. Though we seek to treat general systems of this type, there are a few simplifications we will make based on the system we seek to treat. These simplifications will be pointed out as they occur.

The general form of the equations we wish to solve will then be

$$\nabla X + \mathbf{P}(z, \bar{z})X = \mathbf{W}(z, \bar{z}), \tag{A3.1}$$

where $\nabla = \hat{z}\partial + \hat{\bar{z}}\bar{\partial}$ and $\mathbf{P}$ and $\mathbf{W}$ are some two dimensional vectors of matrices whose elements are generally some known functions of z and $\bar{z}$. We will denote the components of these vectors by P_i and W_i.

We proceed as we would if we had commutivity and seek a general solution to the homogeneous equation $\nabla X + \mathbf{P} = 0$ by proposing a solution of the form[21] $X = e^{F(z,\bar{z})}C$ for some constant matrix C. It is important that we have placed the matrix C on the right as it may not commute with F.

So, we find that the matrix F must satisfy the system

$$\left[\partial F + P_1\right] X = 0, \quad \left[\bar{\partial} F + P_2\right] X = 0. \tag{A3.2}$$

We cannot generally cancel the right hand factor of X since it may not be invertible. This potential failure of invertibility however, cannot come from the factor e^F since it is clearly invertible. Therefore, invertibility can only fail due to our choice in integration constant C. But since this is a constant matrix, we may move to a basis in which C is block diagonal and separate (A3.2) into two case. In the first case, we are in the non-zero block of C and we may cancel off the factor of X. If, however, we are in the zero block of C, then both equations are satisfied trivially and there is no additional constraint on F. But this is precisely what we would expect from the form $X = e^F C$. In the block diagonal basis of C, the values of F which correspond to the zero block of C are completely irrelevant and do not influence the matrix X. We if we do perform cancelation in (A3.2) and apply the condition to the entire matrix F, then we have lost no generality in the finally solution X.

[21] In general, the matrix exponential here would need to be path ordered. However, as we shall see, F may be freely chosen to be a polynomial in the cyclic matrix M defined by (4.2.4), and so will commute with itself for all $z, \bar{z}$. Thus, we may neglect the path order to simplify our analysis.

So, it then follows from the first equation of (A3.2) that F must have the form

$$F(z,\bar{z}) = -\int^{z} P_1(z',\bar{z})\mathrm{d}z' + K(\bar{z}) \tag{A3.3}$$

for some unknown matrix valued function $K(\bar{z})$. But since F must also satisfy the second equation of (A3.2),

$$-\int^{z} \bar{\partial} P_1(z',\bar{z})\mathrm{d}z' + \bar{\partial}K(\bar{z}) + P_2 = 0, \tag{A3.4}$$

so we find

$$K(\bar{z}) = -\int^{\bar{z}} P_2(z,\bar{z}')\mathrm{d}\bar{z}' + \int^{\bar{z}}\int^{z} \bar{\partial}' P_1(z',\bar{z}')\mathrm{d}z'\mathrm{d}\bar{z}'. \tag{A3.5}$$

Therefore, the function F must have the form

$$F(z,\bar{z}) = -\int^{z} P_1(z',\bar{z})\mathrm{d}z' - \int^{\bar{z}} P_2(z,\bar{z}')\mathrm{d}\bar{z}' + \int^{\bar{z}}\int^{z} \bar{\partial}' P_1(z',\bar{z}')\mathrm{d}z'\mathrm{d}\bar{z}'. \tag{A3.6}$$

The integrability condition for these integrations to be well defined is the condition $\partial K(\bar{z}) = 0$ which implies $\int^{\bar{z}} \left(\bar{\partial}' P_1 - \partial P_2\right) \mathrm{d}\bar{z}$. But since we must be free to take this integration over any line in $\bar{z}$, the integrand must be zero. Thus,

$$\bar{\partial} P_1 = \partial P_2. \tag{A3.7}$$

If this condition is satisfied, then (A3.6) represents a general solution to the homogeneous equation.

Returning our attention to the inhomogeneous equations, we will seek a particular solution with the form $X = e^F K(z,\bar{z})$ where F is given by (A3.6) and K is some matrix function. Substituting this ansatz into (A3.1), we then find

$$(\nabla F)\, e^F K + e^F \nabla K + \mathbf{P} e^F K = \mathbf{W}. \tag{A3.8}$$

Therefore,

$$[\nabla F + \mathbf{P}]\, X + e^F \nabla K = \mathbf{W}. \tag{A3.9}$$

But since F is given by (A3.6), the first term must be zero. This follows even though K is not a constant like C was since we argued that there was nothing to be gained by not using cancelation in (A3.2), and so $[\nabla F + \mathbf{P}]$ was required to be zero. Thus, we find that

$$\nabla K = e^{-F}\mathbf{W}. \tag{A3.10}$$

The z component of this equality may be integrated as before to obtain

$$K = \int^z e^{-F(z',\bar{z})} W_1(z',\bar{z}) \mathrm{d}z' + C(\bar{z}) \tag{A3.11}$$

for some function $C(\bar{z})$. The $\bar{z}$ component of (A3.10) then yields

$$\bar{\partial} C(\bar{z}) = e^{-F(z,\bar{z})} W_2 - \bar{\partial} \int^z e^{-F(z',\bar{z})} W_1(z',\bar{z}) \mathrm{d}z'. \tag{A3.12}$$

Integrating this, we find that

$$\begin{aligned} K(z,\bar{z}) = &\int^z e^{-F(z',\bar{z})} W_1(z',\bar{z}) \mathrm{d}z' + \int^{\bar{z}} e^{-F(z,\bar{z}')} W_2(z,\bar{z}') \mathrm{d}\bar{z}' \\ &- \int^{\bar{z}} \bar{\partial}' \int^z e^{-F(z',\bar{z}')} W_1 \mathrm{d}z' \mathrm{d}\bar{z}' \end{aligned} \tag{A3.13}$$

under the integrability condition $\partial(e^{-F} W_2) = \bar{\partial}(e^{-F} W_1)$. If it is the case that $[F(z,\bar{z}), F(z',\bar{z}')] = 0$, as it will be in our case since F will be a polynomial in M, then this condition simplifies to

$$P_1 W_2 + \partial W_2 = P_2 W_1 + \bar{\partial} W_1, \tag{A3.14}$$

where we have used $\partial F = -P_1$ and $\bar{\partial} F = -P_2$.

The full solution to (A3.1) then has the form $X = e^F \left(K(z,\bar{z}) + C \right)$ for constant matrix C and $K(z,\bar{z})$ given by (A3.13) under the assumption that $\mathbf{P}$ and $\mathbf{W}$ satisfy the integrability conditions (A3.7) and (A3.14).

A4 Cyclic Exponential Theory

To begin this presentation, we will suppose that G is a $\mathbb{Z}_n$ representation generated by the object X with identity I. We then seek to describe the object $e^{\alpha X}$. We will partition[22] $\mathbb{N}$ into its equivalence classes modulo n, denoting each class by $[j]_n$. Then we may write $e^{\alpha X}$ as

$$e^{\alpha X} = \sum_{k \in \cup_j [j]_n} \frac{(\alpha X)^k}{k!}, \tag{A4.1}$$

where each of the subsums are taken to be in the standard ordering, though since this is an absolutely convergent series, it doesn't much matter.

[22] We use the convention which counts zero as an element of $\mathbb{N}$.

Now, we reorder the sum (A4.1) to write the finite sum

$$e^{\alpha X} = \sum_{j=0}^{n-1} X^j \mu_j^n(\alpha), \tag{A4.2}$$

where we have defined the function $\mu_j^n(\alpha)$ by

$$\mu_j^n(\alpha) = \sum_{k \in [j]_n} \frac{\alpha^k}{k!}. \tag{A4.3}$$

It is important to point out that these special functions are not completely unfamiliar. If it were the case that $n = 4$, then we have described the exponential of the imaginary unit. It follows immediately that

$$\begin{aligned} \sin(\alpha) &= \mu_1^4(\alpha) - \mu_3^4(\alpha), \\ \cos(\alpha) &= \mu_0^4(\alpha) - \mu_2^4(\alpha), \end{aligned} \tag{A4.4}$$

so in a very real sense, these μ functions are just generalizations of the sine and cosine functions. It is likely that these functions will also have nice properties and symmetries like the trigonometric functions do, however, we will not develop the theory of these functions any further.

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