COUNTING SOLUTIONS OF $a^2 + pbc = 0$ IN A CUBE

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ABSTRACT. For a prime p, let $s_p(n)$ be the number of integer triples (a, b, c) which satisfy $a^2 + pbc = 0$, where a, b, c are bounded by natural number n, and p is prime. Some sequences of this form have had limited numbers of terms contributed to the OEIS, while others have had no contributions at all. A non-recursive, generalized algorithm was theorized and developed, to produce the first n terms of the sequence relating to the equation $a^2 + pbc = 0$.

1. INTRODUCTION

Clark Kimberling has contributed a substantial number of sequences to the Online Encyclopedia of Integer Sequences (OEIS), some of which involve counting solutions to various equations. In this paper, we consider equations of the form

$$a^2 + pbc = 0,$$

where p is a prime. These equations each have three variables, with each variable being bound by n in absolute value. In other words, for variables a, b, c:

$$-n \leq a \leq n$$
$$-n \leq b \leq n$$
$$-n \leq c \leq n.$$

Definition. For a given prime p, for each natural number let $n \geq 0$, $s_p(n)$ to be the number of integer triples (a, b, c) which satisfy $a^2 + pbc = 0$, where a, b, c are bound by n. The s_p *sequence for a given* p, then, is: $s_p(0), s_p(1), s_p(2), s_p(3), \ldots$.

The first 5 sequences of this form, relating to the first 5 primes, have been contributed to the OEIS at the following links:

p	OEIS sequence
2	A211423
3	A211424
5	A334524
7	A334525
11	A334526

Remark 1. Previous algorithms to find the nth term of the s_p sequence, for a given p, involved generating $(2n + 1)^3$ triples. For large n, computing $s_p(n)$ requires extensive time with this algorithm. As a result, sequences counting solutions to equations of the form $a^2 + pbc = 0$ had limited terms listed on the OEIS. In this paper, we develop and implement a faster algorithm to generate the first n terms of the s_p sequence.

Date: August 28, 2020.

After code based on the algorithm described in this paper was contributed to sequences on the OEIS, two authors contributed their own code. David A. Corneth contributed code using PARI to sequence A211423, while Robert Israel contributed Maple code to sequences A334524 and A334525. In Section 5.5, we discuss these alternate approaches.

2. Initial Observations

In this section, we consider the s_2 sequence, which counts solutions to the equation $a^2+2bc=0$. The first 10 terms of $s_2(n)_{n\geq 0}$ are $1, 5, 17, 21, 33, 37, 49, 53, 73, 85$.

Definition. A triple (a,b,c) that satisfies the equation $a^2+pbc=0$ for some p is a *solution*. A solution (a,b,c) to the equation $a^2+pbc=0$ for some p is a

$$\textit{new solution at } n \text{ if } \max(|a|,|b|,|c|)=n;$$
$$\textit{returning solution at } n \text{ if } \max(|a|,|b|,|c|)<n;$$
$$\textit{out of bounds solution at } n \text{ if } \max(|a|,|b|,|c|)>n.$$

Example 2. Consider the s_2 sequence. $s_2(0)=1$, since there is exactly one solution to the equation $a^2+2bc=0$ with a,b,c being bound by 0: $(0,0,0)$. This solution is a new solution at 0.

Example 3. The triple $(0,0,0)$ also serves as a solution to the equation $a^2+2bc=0$ when a,b,c are bound by 1. At $n=1$, this solution is a returning solution.

Example 4. The triple $(-2,-2,1)$ also serves as a solution of the equation $a^2+2bc=0$. If our bound is still at $n=1$, we conclude that the solution $(-2,-2,1)$ is out of bounds.

Remark 5. If a triple (a,b,c) satisfies the equation $a^2+pbc=0$ for some p, then the following seven triples will also satisfy the equation: (a,c,b), $(a,-b,-c)$, $(a,-c,-b)$, $(-a,b-c)$, $(-a,c,b)$, $(-a,-b,-c)$, $(-a,c,b)$.

Example 6. Recall the first 10 terms of the s_2 sequence. For $n=2$, 12 new solutions to the equation $a^2+2bc=0$ are observed. The new solutions are as follows:

$$\{(0,2,0),(0,-2,0),(0,0,2),(0,0,-2),(-2,-2,1),(-2,2,-1),$$
$$(-2,-1,2),(-2,1,-2),(2,-2,1),(2,2,-1),(2,-1,2),(2,1,-2)\}.$$

The first 4 new solutions each contain a pair of 0s and a 2, with alternating signs. The remaining 8 solutions all have a pair of 2s and a 1, with alternating signs. These two collections of triples illustrate *families of solutions.*

Definition. A *family of solutions* is the set of triples $\{(a,b,c)$, (a,c,b), $(a,-b,-c)$, $(a,-c,-b)$, $(-a,b-c)$, $(-a,c,b)$, $(-a,-b,-c)$, $(-a,c,b)\}$, so long as one triple satisfies the equation $a^2+pbc=0$, for some p. The *family value* is defined as $\max(|a|,|b|,|c|)$.

Definition. A solution is a *primitive solution* if $\gcd(a,b,c)=1$. A *primitive family of solutions* is a family of solutions containing a primitive solution.

Remark 7. If (a,b,c) is a new solution at some n to the equation $a^2+pbc=0$, for some p, then $(\beta a,\beta b,\beta c)$ is a new solution at βn, for every positive integer β.

Remark 8. If (a, b, c) is a primitive solution at some n to the equation $a^2+pbc = 0$, for some p, then for every $\beta \geq 2$, $(\beta a, \beta b, \beta c)$ is not a primitive solution.

Remark 9. The primitive family of solutions with the least family value is $\{(0, 1, 0),$ $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)\}$. This family of solutions, along with all other families of solutions of the form $\{(0, \beta, 0)$, $(0, -\beta, 0)$, $(0, 0, \beta)$, $(0, 0, -\beta)\}$, for some non-zero β, are the only families of solutions with exactly 4 distinct members.

If a family of solutions does not have exactly 4 distinct members, it either has exactly 8 distinct members, or is the *trivial family of solutions* $\{(0, 0, 0)\}$.

Proposition 10. *Let p be a prime number, and let $n \geq 0$. The number of new solutions at n to the equation $a^2 + pbc = 0$ is a multiple of 4, and not a multiple of 8.*

Proof. Solutions to our equation $a^2+pbc = 0$ only appear in a family of 4 solutions, or a family of 8 solutions, obtained from a primitive family. Thus, we can represent the total of new solutions at a given n as $4 + 8m$, where m denotes the number of families of solutions at that n. We conclude, therefore, that since m is a natural number, the number of new solutions at a given n is a multiple of 4, and not a multiple of 8. □

3. Viewing $s_p(n)$ as a Discrete Sum

Lemma 11. *Let p be a prime number, and let $n \geq 0$. Given solution (a^*, b^*, c^*) to equation $a^2 + pbc = 0$, and natural number α, such that α divides a^*, b^*, c^*, triple $(\frac{a^*}{\alpha}, \frac{b^*}{\alpha}, \frac{c^*}{\alpha})$ is a solution to equation $a^2 + pbc = 0$.*

Proof. We have $a^{*2} + pb^*c^* = 0$. Divide both sides by α^2. We then have $(\frac{a^*}{\alpha})^2 + p(\frac{b^*}{\alpha})(\frac{c^*}{\alpha}) = 0$, concluding that $(\frac{a^*}{\alpha}, \frac{b^*}{\alpha}, \frac{c^*}{\alpha})$ is a solution. □

Definition. HI

Theorem 12. *Every family of solutions can be generated, via greatest common divisor, from a primitive family of solutions.*

Proof. Consider the arbitrary family of solutions containing (a^*, b^*, c^*). Either $\gcd(a^*, b^*, c^*) = 1$ or $\gcd(a^*, b^*, c^*) > 1$. If $\gcd(a^*, b^*, c^*) = 1$, then by definition, the family of solutions is primitive. If $\gcd(a^*, b^*, c^*) > 1$, let $\gcd(a^*, b^*, c^*) = \beta$. The triple $(\frac{a^*}{\beta}, \frac{b^*}{\beta}, \frac{c^*}{\beta})$, by Lemma 11, is also a solution. The family of solutions containing (a^*, b^*, c^*), then, is generated by multiplying the primitive family of solutions containing $(\frac{a^*}{\beta}, \frac{b^*}{\beta}, \frac{c^*}{\beta})$ by β. □

We have shown that new solutions at a given n only appear in the trivial family of solutions, or in families with exactly 4 or 8 members. We also know that each family of solutions is generated from a particular primitive family. If we can generate all primitive families of solutions, then we can generate all families of solutions. With that information, we can calculate $s_p(n)$.

We know that all solutions (a, b, c) are counted for the first time at its family's value. To know the total number of solutions which satisfy the equation $a^2+pbc = 0$ at a given n, we need to figure out all new solutions that appear at each natural number $m \leq n$. With this is mind, we define the following.

Notation. For a prime p, let $f_p(m)$ be the number of triples (a, b, c) which satisfy the equation $a^2 + pbc = 0$, with $\max(|a|, |b|, |c|) = m$. Then,

$$s_p(n) = \sum_{m=0}^{n} f_p(m).$$

As a brief example, to find $s_2(3)$, we find all new solutions which appear at $m = 0, 1, 2, 3$, and sum them to calculate the total number of solutions to the equation $a^2 + 2bc = 0$, when our variables are bound by $n = 3$:

$$s_2(3) = f_2(0) + f_2(1) + f_2(2) + f_2(3) = 21.$$

With this definition, we consider $s_p(n)$ as a discrete sum, where each $f_p(m)$ equals the number of solutions with appropriate bounds with a family value m.

4. Viewing $f_p(m)$ as a Discrete Sum

In this section, we consider $f_p(m)$ as a sum of families of solutions. As stated in Theorem 12, all families of solutions can be generated via greatest common divisor from a primitive family of solutions. To view all solutions with family value m, then, we can simply find all primitive families of solutions with family values which divide m. For all primitive families found at these divisors, we know there will be a family of 8 solutions, with the exception of the trivial family to be found at divisor $d = 1$. With all of this being said, we define:

Notation. For a prime p, let $g_p(d)$ be the number of primitive families with family value d. Therefore,

$$f_p(m) = -4 + 8\sum_{d|m} g_p(d).$$

As a brief example, to find $f_2(3)$, we must compute $g_2(d)$, for all divisors d of 3. The only divisors of 3 are itself and 1: 3 does not serve as a family value of any primitive families, and 1 serves as the family value of the family containing the triple $(0, 1, 0)$. So then, $g_2(3) = 0$ and $g_2(1) = 1$, meaning $8\sum_{d|3} g_2(d) = 1$. Therefore, $f_2(3) = -4 + 8\sum_{d|3} g_2(3) \to f_2(3) = -4 + 8(1) = 4$.

Since $s_p(n)$ is dependent on $f_p(m)$, and since $f_p(m)$ is dependent on $g_p(d)$, we know to find $s_p(n)$, we need to know how to find primitive families at d values. primitive families will be discussed in the remaining theorems and lemmas of this section.

Theorem 13. *If (a, b, c) is a primitive family for some primitive family value d, with $a = d$, then d is of the form $p^i j$, where $i \geq 1$ and $j \geq 1$ such that $j \nmid p$.*

Proof. We see that the right side of the equation $a^2 = -pbc$ is divisible by p, there must be at least one p included in a. The number a, then, must have some number of p's.

Now, assume a is of the form p^i. Taking our equation:

$$a^2 = -pbc.$$

We know that since a is a collection of i p's, and the right side of our equation is divisible by p, then $bc = p^{2i-1}$. Further, since a is the maximum value, we know that $b, c \leq p^i$. We also can check that the triple (p^i, p^i, p^i) will not serve as a solution, so at most one of b and c may equal p^i. By doing this, we know that each of a, b, c is divisible by p. We now have $\gcd(a, b, c) \geq p \neq 1$, which means that (a, b, c) is no longer a primitive family. This is a contradiction, and we deduce that a cannot be of the form p^i. The only remaining possibility for a, then, is to be $p^i j$. □

Lemma 14. *If (a, b, c) is a primitive family for some primitive family value d, with $b = d$ or $c = d$, and $p \nmid d$, then d is of the form j^2: a square, which we know is not divisible by p.*

Proof. Either b or c is our d value, which serves as our maximal element. Without loss of generality, assume b is our maximal element: $b = d$. It is given that $p \nmid b$. Assume, toward contradiction, that b is not a square. Then, for some prime to some power q^j in the prime factorization of b, j is odd. We know:

$$a^2 = -pbc.$$

Our left side of the equation needs to have an even number of q's, as the left side is a perfect square. So, in order to arrive at an even number of q's on the right side, we conclude that c must have at least one q in its prime factorization. Then, we have $\gcd(a, b, c) \geq q \neq 1$, which means (a, b, c) is not a primitive family. We deduce that our assumption of b being not a square, is incorrect. So, our maximal element $b = d$ is a square without p, if $p \nmid d$. □

Theorem 15. *If (a, b, c) is a primitive family for some primitive family value d, with $b = d$ or $c = d$, then d is of either the form j^2 (a square without p) or the form $p^{2i-1} j^2$ (a square, times p).*

Proof. Either b or c is our d value, which serves as our maximal element. Without loss of generality, assume b is our maximal element. The number b, clearly, either is or is not divisible by p. By Lemma 14, if $p \nmid b$, then b is a square without p.

We seek to prove: if $p \mid b$, then it is of the form $p^{2i-1} j^2$ (a square, times p). We know that $p \mid b$. Assume, toward contradiction, that there are an even number of p's in b's prime factorization– call this even number $2i$. Recall:

$$a^2 = -pbc.$$

We know that there are at least $2i+1$ p's on the right side of the equation. Since the left side is a square, we know there must be an even number of primes on the left side, and therefore, the right. The only way to do this is to deduce c is divisible by p^k, for some odd k. But, that leads to $\gcd(a, b, c) \geq p \neq 1$, a contradiction. So, if $p \mid b$, then there must be an odd number of primes in its prime factorization.

Now assume, once again toward contradiction, that the remainder of p's prime factorization does not produce a perfect square. Then, there exists a prime $q \neq p$ in $\frac{b}{p^{2i-1}}$ such that there are an odd number of them in the prime factorization of b. Then, in a similar line of argument to our previous contradiction, we observe:

$$a^2 = -pbc.$$

If there are an odd number of q's in the prime factorization of p, and $q \neq p$, then to obtain an even number of q's on the right side to satisfy the squareness of the left side, we deduce $\frac{b}{q^k} \in \mathbb{Z}$ for some odd k, which leads us to the contradiction $\gcd(a, b, c) \geq q \neq 1$.

We conclude, therefore, that if (a, b, c) is a primitive family for some primitive family value d, with $b = d$ or $c = d$, then d is of either the form j^2 (a square without p) or the form $p^{2i-1}j^2$ (a square, times p). The expression $\frac{b}{p^{2i-1}}$, the remaining piece of d, therefore, must be square.

We conclude that b is of either the form j^2 (a square without p) or the form $p^{2i-1}j^2$ (a square, times p). □

Theorem 16. *All primitive family values take one of the following forms (for $i \in \mathbb{Z}; i \geq 1, p \nmid j$):*

1. $p^i j$ *(some number of p's, times j)*
2. j^2 *(a square without p)*
3. $p^{2i-1}j^2 = p(p^{i-1}j)^2$ *(a square, times p)*

Proof. All primitive families can be identified by their primitive family value. If the primitive family value occurs as the first element of the triple, by Theorem 13, we know it takes the first form listed. If the primitive family value occurs as the second or third element of our triple, by Theorem 15, we know it takes either the second or third form listed. □

5. Algorithm for Computing $s_p(n)$

If we can accurately identify primitive family values, then we know where each primitive family, and consequently, where each family of solutions is located. We can then construct the s_p sequence at a given prime in a far more efficient way. Rather than generate every possible triple, as previous algorithms did, we can use primitive family values to generate far fewer triples. Again, recall our algorithm:

$$s_p(n) = \sum_{m=0}^{n} f_p(m).$$

For a prime p, let $f_p(m)$ be the number of triples (a, b, c) which satisfy the equation $a^2 + pbc = 0$, with $\max(|a|, |b|, |c|) = m$. Therefore,

$$f_p(m) = -4 + 8 \sum_{d|m} g_p(d),$$

where $g_p(d)$ is the number of primitive families with family value d.

The crucial piece of the algorithm, then, relies on calculating the number of primitive families with a specified family value. To do this, three "case" algorithms were written: one to test for each form a primitive triple may take. These algorithms begin by identifying if d is of the form being tested, and if so, triples including d in the appropriate spot are constructed to test if primitive families exist there.

5.1. **Case 1.** We want an algorithm to check whether a given d took the form $p^i j$ (some number of p's, times some non-p), and if this is the case, to construct intelligent triples including d which may be primitive.

Recall, if d is to be of the form $p^i j$, then it will appear as the a term of our triple, by Theorem 13. At this point, we will use the notation $\prod q^e$: d as a product of primes q, raised to exponents e. As an example, for $d = 30$, $d = \prod q^e = 2^1 \times 3^1 \times 5^1$. Recall:

$$a^2 = -pbc.$$

So, since a is squared, and we know $a = \prod q^e$, then the right side of the equation needs to equal $(\prod q^e)^2$. But, since p already divides the right side of the equation, we can deduce that $bc = \frac{(\prod q^e)^2}{p}$.

Using our examples of $d = 30$, we know $bc = \frac{900}{2} = 450$. Note: if one prime p_i in $\frac{(\prod q^e)^2}{2}$ is to be included in, say, our b term, then *all copies of p_i found in $\frac{(\prod q^e)^2}{2}$ must be in our b term as well.* Otherwise, $\gcd(a, b, c) \geq p_i \neq 1$, which is impossible because (a, b, c) is a primitive family.

Theorem 17. *If there are k distinct primes in the prime factorization of d, then there are 2^{k-1} possible pairs of b, c to check as potential pieces to primitive families.*

Proof. We know there are k types of primes included in our prime factorization of d. For our b term, there can either be none of these primes (in this case, b would equal 1, and c would have all primes which divide c in $(\prod q^e)^2$), or there can be one of these primes, or two, or as many as k primes (all of them, then c would equal 1). To find the total number of possibilities of choices of primes in our b term:

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k} = \sum_{i=0}^{k} \binom{k}{i} = 2^k.$$

Once our b is determined, there is exactly one choice for our c value, leaving 2^k different triples to choose from. But, since b and c are communicative, there are exactly half of the total number of triples to check: 2^{k-1}. Recall, if the pair (b, c) is permitted to be in our triple with a, then so will the triple (c, b), as they are in the same family of solutions. We conclude, then, that for k distinct primes, there are exactly 2^{k-1} pairs (b, c) to check as possible triples with a. □

Now that we have constructed these 2^{k-1} triples, we plug each of them into our equation $a^2 + pbc = 0$, along with $a = d$, and we accept a pair if $\max(|a|, |b|, |c|) = a = d$. Each accepted pair, combined with $a = d$, constitutes a new primitive family at d.

Example 18. Using our $d = 30$ example, for $g_2(30)$ the primitive family which satisfies our equation is $(-30, -25, 18)$. Our algorithm will run for $d = 30$ as follows:

1. Check if d is the product of some 2's and a non-2. In this case, 30 is of this form, so we continue on.
2. Calculate what $\prod q^e$ will be. In this case, $\prod q^e = 2^1 \times 3^1 \times 5^1$.
3. Take all possible combinations of each kind of prime, to be placed in either b or c. In the case of $d = 30$, we have three distinct primes, leading to $2^{3-1} = 2^2 = 4$ triples to check: $(1, 450), (2, 225), (9, 50), (25, 18)$.
4. For each pair (b, c) constructed, we have the triple (a, b, c), where $a = d$. In this case, the only pair (b, c) which has $\max(|a|, |b|, |c|) = a$ is $(25, 18)$. We conclude, then, that for $d = 30$ that one primitive family appears: the primitive family including the triple $(-30, -25, 18)$.

The above algorithm, as a reminder, is designed to tell us the number of primitive families which appear for the first time at $d = 30$. The algorithm would conclude, then, by saying that in this case, one new primitive family appears at d.

5.2. **Case 2.** We want an algorithm to check whether a given d took the form j^2 (a square, without p), and if this is the case, to construct intelligent triples including d which may be primitive.

Theorem 19. *For any primitive family (a, b, c), with b as the maximal element $(b = d)$ and $p \nmid b$, then a is divisible by both p and $b^{1/2}$ in its prime factorization. Further, if a has i p's in its prime factorization, c will have $2i - 1$ p's in its prime factorization.*

Proof. Assume triple (a, b, c) is a primitive family at primitive family value d, with $b = d$. By Lemma 14, since b is the maximal element and $p \nmid b$, b is a square wihtout p. Recall:

$$a^2 + pbc = 0 \to a^2 = -pbc.$$

We have $p \mid a^2$, which implies $p \mid a$.

The number b has some prime factorization, where each distinct prime is raised to an even power, since b is a square. To maintain the equality of the equation, a must be disible by half of each number of all primes which divide b; otherwise, the prime factorizations of a^2 and pbc would not be able to match. We conclude, then, that a takes the form $p^i b^{1/2}\gamma$, for some $i, \gamma \geq 1$.

Assume a has i p's in its prime factorization. In the above equation, then, the left side of the equation is divisible by p^l, for some even l. So, since the left and right sides are equal, $-pbc$ must also be divisible by exactly $2i$ p's. We know b is not divisible by p none, since $p \nmid b$. The remaining $2i - 1$ p's, then, are found in the c term of the triple. The number c, then, takes the form $p^{2i-1}\delta$ for some $\delta \geq 1$. □

In Theorem 19, we introduce more new notation: γ and δ. By our conclusions drawn in Theorem 19, we find the result:

$$a^2 = -pbc \to (p^i b^{1/2}\gamma)^2 = -pb(p^{2i-1}\delta) \to \gamma^2 = \delta.$$

We know that for (a, b, c) to be a primitive family, $\gcd(a, b, c) = 1$ needs to be true. We have shown that any prime included in our b term will appear evenly, and that $b^{1/2}$ is included in our a term, meaning none of these primes can be included in our c term. We also know that if a prime is to be included in both b and c, it would need to be included in a as well, which would make $\gcd(a, b, c) = 1$ a false statement. So, if a prime is in b, it must be in a and must not appear in c. If a prime were to appear in c, then in a similar fashion, it also must appear in a and must not appear in b.

We have assumed that our d is appearing as our b term for this argument. If we allow for other primes to be included in our a and c terms, though, these make for valid primitive families at $\max(|a|, |b|, |c|)$. Should $\max(|a|, |b|, |c|) = b$, then we have a primitive family to be counted at d otherwise, $\max(|a|, |b|, |c|) > d$, and we reject. These primes in a and c which aren't in b are introduced by our γ and δ terms.

Example 20. Our algorithm for Case 2, then, is as follows (we will use $d = 49$ as an example for $g_2(49)$):

1. Check if d is a square not divisible by 2. In this case, 49 is of this form, so we continue on.
2. We know exactly what our b term can be. We set our a term to have a minimum value of $pb^{1/2}\gamma$ and our c term to have a minimum value of $p\delta$,

for $\gamma, \delta = 1$. For $d = 49, p = 2$, our a term needs to have a minimum of 14, and our c term needs to have a minimum of 2.

3. At this point, we already have one primitive family: $(pb^{1/2}, b, p)$ (for $d = 49$, we have $(14, -49, 2)$).
4. We know that γ and δ must be products exclusively of primes which are not included in the prime factorization of b. We increment γ by one, which also increments δ. We accept the triple $(pb^{1/2}\gamma, b, p\delta^2)$ if both of the following are true:
   - $\gcd(a, b, c) = 1$ (if γ and δ are divisible by primes not included in b)
   - $\max(|a|, |b|, |c|) = b$
5. Repeat the incrementation described in Step 4 until $\max(|a|, |b|, |c|) \neq b$. At this point, we would have now been looking at primitive families defined by primitive family values higher than d. For $d = 49, p = 2$, we increment γ and δ, and we also accept the triples $(28, -49, 8)$ and $(42, -49, 18)$. Incrementing to $\gamma = 4$ yields the triple $(56, 49, 32)$. This is a primitive family, but not for $d = 49$. We reject this primitive family and conclude.

This algorithm, like Case 1, returns the number of primitive families which appear for the first time at d. At $d = 49, p = 2$, then, we have three new primitive families which arise, and conclude.

5.3. **Case 3.** We want an algorithm to check whether a given d took the form $p^{2i-1}j^2$ (a square, times p), and if this is the case, to construct intelligent triples including d which may be primitive. This algorithm is paralleled by Case 2, given their similar structures. The form $p^{2i-1}j^2$, after all, is just a square, times an odd number of p's. Once again, we recall:

$$a^2 = -pbc.$$

We know that a^2 must contain all the primes in b to make the equation equal, and since there are $2i-1$ p's in b's prime factorization, there are $2i$ p's which divide the right side, so a must have i p's. Since b is of the form $p^{2i-1}j^2$ (a square, times p), a is minimized to the value $p^i(b/p^{2i-1})^{1/2}$, and b is minimized to 1, as there is no restriction other than not having primes which are found in c. We re-introduce our γ and δ notation, and use the parallel algorithm.

Example 21. Here, we will start with $g_2(72)$):

1. Check if d is of the form p times a square. 72 fits this classification.
2. We know exactly what our c term can be. We set our a term to have a minimum value of $pc^{1/2}\gamma$ and our b term to have a minimum value of δ, for $\gamma, \delta = 1$.
3. At this point, we already have one primitive family: $(pc^{1/2}, 1, c)$.
4. We know that γ and δ must be products exclusively of primes which are not included in the prime factorization of b. We increment γ by one, which also increments δ. We accept the triple $(pc^{1/2}\gamma, \delta^2, c)$ if both of the following are true:
   - $\gcd(a, b, c) = 1$ (if γ and δ are divisible by primes not included in c)
   - $\max(|a|, |b|, |c|) = c$
5. Repeat the incrementation described in Step 4 until $\max(|a|, |b|, |c|) \neq c$. At this point, we would have now been looking at primitive families defined by primitive family values higher than d.

Given the similarity in structure between the forms checked by Case 2 and Case 3, their algorithms being almost identical is not surprising.

5.4. **Case Algorithms–A Final Word.** The three case algorithms, as shown, were developed to calculate $g_p(d)$: the number of primitive families which appear at divisor d of m, using prime p in our equation. Now that we can calculate where all primitive families occur, and how many appear at each divisor of m, we know the number of families of solutions which appear for the first time at m, and with that, the total number of new solutions at any m:

For a prime p, let $f_p(m)$ be the number of triples (a, b, c) which satisfy the equation

$$a^2 + pbc = 0, \text{ with } \max(|a|, |b|, |c|) = m.$$

$$f_p(m) = -4 + 8sum_{d|m} g_p(d).$$

Using this information, we now know all solutions which appear at any $m \leq n$, and with that, we can calculate the total number of solutions which appear at $s_p(n)$:

$$s_p(n) = sum_{m=0}^{n} f_p(m).$$

5.5. **Alternate Algorithm Discussion.** David A. Corneth contributed code written in PARI to the s_2 sequence, which can be found on the OEIS under sequence number A211423. He made two separate contributions, with the second being more efficient than the first. Corneth's approach, like the approach described in this paper, involves constructing the s_2 sequence via the difference seqeunce. Corneth's approach is unique, and involves a sort of distinction between new solutions via primitive families and via nonprimitive families. His algorithm, though, relies on divisors in terms of their magnitude when related to other divisors; the algorithm described here does not rely on this notion.

Robert Israel contributed code written in Maple to the s_5 and s_7 sequences, which can be found at sequences A334524 and A334525, respectively. Israel's approach also relies on the difference sequences. Israel's approach does not explicitly calculate each member of a given triple, while the algorithm described here does.

It should be noted that reading Corneth's and Israel's work was refreshing, and is recommended to the reader.

6. Concluding Remarks

This algorithm relies on the generation of far fewer triples than previous algorithms generated. With this in mind, this algorithm developed was able to produce the first 1,000,000 terms of the s_2 sequence in 61.682 seconds, when only the first 257 terms were known previously. Of the first million terms, 20,000 have been contributed to the OEIS. The s_p sequence, evidently, was a project which dually contributes terms to defined and not yet defined sequences on the OEIS. Sequence $s_p(n)$ gives the nth term of the sequence relating to the number of solutions to the equation $a^2 + pbc = 0$, for some prime.

6.1. **Acknowledgements.** Completing this project was the most memorable academic endeavor of my undergraduate career. I would like to thank Dr. Eric Rowland, whose advisement and insight proved crucial to the development of the project. This project would not be what it is without his contribution.

References

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